P.V. numbers

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Theorem 1. *Let R be an integral domain with field of fractions K. Let $p \in R[x]$ be a monic polynomial of degree d with roots $r_1, \ldots, r_d \in \overline{K}$. Then $r_1^k + \ldots + r_d^k \in R$ for all $k \geq 0$.*

Proof. Let C be the $d \times d$ companion matrix of p. Then the roots of p are given by the eigenvalues of C which are given by the diagonal entries of the Jordan normal form $J = P^{-1}CP$ of C. By reindexing the roots of p, we may assume without loss of generality that $J_{ii} = r_i$ for all $1 \leq i \leq d$. Since J is upper triangular, $(J^k)_{ii} = (J_{ii})^k$ for all $1 \leq i \leq d$. Then

$$\sum_{i=1}^{d} r_i^k = \sum_{i=1}^{d} (J_{ii})^k = \sum_{i=1}^{d} (J^k)_{ii} = \mathrm{tr}(J^k) = \mathrm{tr}((P^{-1}CP)^k) = \mathrm{tr}(P^{-1}C^kP) = \mathrm{tr}(C^k) = \sum_{i=1}^{d} (C^k)_{ii}.$$

However, the entries of C lie in R so the entries of C^k also lie in R. □

For $x \in \mathbb{R}$, let $\|x\|_{\mathbb{Z}}$ denote the distance from x to the nearest integer.

Corollary 1. *Let $p \in \mathbb{Z}[x]$ be a monic polynomial of degree d with roots $r_1, \ldots, r_d \in \mathbb{C}$. If $|r_i| < 1$ for all $2 \leq i \leq d$ then $r_1 \in \mathbb{R}$ and $\left\|r_1^k\right\|_{\mathbb{Z}} \to 0$ as $k \to \infty$. More precisely,*

$$\left\|r_1^k\right\|_{\mathbb{Z}} \leq (d-1)\left(\max_{2 \leq i \leq d} |r_i|\right)^k$$

for all $k \geq 0$. Here the maximum is taken to be 0 if $d = 1$.

Proof. By dividing out by a suitable power of x, we may assume without loss of generality that $p(0) \neq 0$. If $d = 1$ then $p = x - r_1$ so $r_1 \in \mathbb{Z}$ and the result follows. Now suppose that $d \geq 2$. Then

$$1 \leq |p(0)| = |r_1 \ldots r_d| = |r_1| \cdot \ldots \cdot |r_d| < |r_1|$$

so $|r_1| > 1$. Since p has real coefficients, $p(\overline{r_1}) = \overline{p(r_1)} = 1$ so $\overline{r_1} \in \mathbb{C}$ is a root of p with $|\overline{r_1}| = |r_1| > 1$. Then $\overline{r_1} = r_1$ so $r_1 \in \mathbb{R}$. Theorem 1 gives that

$$\left\|r_1^k\right\|_{\mathbb{Z}} \leq \left|r_1^k - \sum_{i=1}^{d} r_i^k\right| = \left|\sum_{i=2}^{d} r_i^k\right| \leq \sum_{i=2}^{d} |r_i|^k \leq (d-1)\left(\max_{2 \leq i \leq d} |r_i|\right)^k$$

as desired. □