Mean Value Theorem for Integrals

Thomas Browning

November 2017

Recall the statement of Problem 4.2.7 in Folland's *Advanced Calculus*.

Theorem 1 (Problem 4.2.7 in Folland's *Advanced Calculus*)**.** *Let $\varphi\colon [a,b] \to \mathbb{R}$ be C^1 and increasing on $[a,b]$ and let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then there exists a $c \in [a,b]$ such that*

$$\int_a^b f(x)\varphi(x)dx = \varphi(a)\int_a^c f(x)dx + \varphi(b)\int_c^b f(x)dx.$$

The suggested proof uses integration by parts on the left integral and then applies Theorem 4.24 in the text. This requires the continuity of φ'. It is then natural to ask whether this result can be proved without Theorem 4.24 so that the condition that φ be C^1 could be dropped. The proof of the following result avoids Theorem 4.24 and thus greatly weakens the assumptions of φ and f.

Theorem 2 (The Mean Value Theorem for Integrals)**.** *Let $\varphi\colon [a,b] \to \mathbb{R}$ be monotone and let $f\colon [a,b] \to \mathbb{R}$ be integrable. Then there exists a $c \in [a,b]$ such that*

$$\int_a^b f(x)\varphi(x)dx = \varphi(a^+)\int_a^c f(x)dx + \varphi(b^-)\int_c^b f(x)dx \text{ where } \varphi(a^+) = \lim_{x\to a^+}\varphi(x),\ \varphi(b^-) = \lim_{x\to a^-}\varphi(x).$$

Proof. By replacing φ by $\pm\varphi$, we may assume without loss of generality that φ is decreasing. By replacing φ with $\varphi - \varphi(b^-)$, we may assume without loss of generality that $\varphi(b^-) = 0$. Then it suffices to show that

$$\int_a^b f(x)\varphi(x)dx = \varphi(a^+)\int_a^c f(x)dx$$

for some $c \in [a,b]$. Now let $\psi\colon [a,b] \to [0,1]$ be a decreasing step function with $0 \le \psi \le \varphi$. Then

$$\psi = \sum_{j=1}^n c_j 1_{[x_{j-1},x_j]}$$

for constants $c_1 > c_2 > \ldots > c_n = 0$ and $a = x_0 < x_1 < \ldots < x_n = b$. Defining $F(t) = \int_a^t f(x)dx$ gives

$$\int_a^b f(x)\psi(x)dx = \sum_{j=1}^n c_j \int_{x_{j-1}}^{x_j} f(x)dx = \sum_{j=1}^n c_j\left(F(x_j) - F(x_{j-1})\right) = c_nF(b) - c_1F(a) + \sum_{j=1}^{n-1} F(x_j)\left(c_j - c_{j+1}\right).$$

However, $c_n = 0$ and $F(a) = 0$ so we can rewrite this equality as

$$\int_a^b f(x)\psi(x)dx = c_1\sum_{j=1}^{n-1} F(x_j)\frac{c_j - c_{j+1}}{c_1}.$$

This sum is a weighted sum of the $F(x_j)$ where the weights are positive and sum to 1. Then the value of the weighted sum must lie between the minimum and maximum of the $F(x_j)$. By the continuity of F, the intermediate value theorem guarantees that this value equals $F(c)$ for some $c \in [a,b]$ so

$$\int_a^b f(x)\psi(x)dx = c_1 F(c) = \psi(a^+)F(c).$$

Now let $\{\psi_j : [a,b] \to \mathbb{R}\}_{j=1}^{\infty}$ be a family of decreasing step functions such that $0 \le \psi_1 \le \psi_2 \le \ldots \le \varphi$ and such that $\psi_j \to \varphi$ pointwise as $j \to \infty$. As an example of such a sequence, one could let ψ_j be given by rounding the value of φ down to the nearest $1/2^j$th. For each $j \ge 1$, let c_j be the constant given by applying the previous argument to ψ_j. By passing to a convergent subsequence, we may assume without loss of generality that the sequence $\{c_j\}_{j=1}^{\infty}$ converges to some $c \in [0,1]$. Then for each $j \ge 1$,

$$\int_a^b f(x)\psi_j(x)dx = \psi_j(a^+)F(c_j).$$

Taking the limit as $j \to \infty$ gives that

$$\int_a^b f(x)\varphi(x)dx = \varphi(a^+)F(c)$$

where interchanging the limit and integral is valid by the dominated convergence theorem. $\square$

This proof highlights a useful real analysis technique that works especially well when dealing with Lebesgue integration: You first prove a result for step functions and then extend the result to all functions by approximating that function by a sequence of step functions.