Dirichlet Series

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1 Convergence of Dirichlet Series

Let $a_1, a_2, \dots$ be a sequence of complex numbers. Consider the partial sums defined by

$$S_n = \sum_{k=1}^{n} a_k.$$

Suppose that there is an $\alpha > 0$ such that $|S_n| \leq Cn^\alpha$ for all $n \geq n_0$. Let K be a compact subset of the half-plane $\mathcal{H}_\alpha = \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$. Then

$$K \subseteq \{s \in \mathbb{C} : \operatorname{Re}(s) \geq \alpha + \varepsilon \text{ and } |s| \leq M\}$$

for some $\varepsilon > 0$ and $M > 0$. If $n_0 \leq n_1 \leq n_2$ and $s \in K$ then

$$\begin{aligned}
\left|\sum_{n=n_1+1}^{n_2} \frac{a_n}{n^s}\right| &= \left|\sum_{n=n_1+1}^{n_2} \frac{1}{n^s}(S_n - S_{n-1})\right| \\
&= \left|\frac{S_{n_2}}{(n_2+1)^s} - \frac{S_{n_1}}{(n_1+1)^s} + \sum_{n=n_1+1}^{n_2} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) S_n\right| \\
&\leq \frac{Cn_1^\alpha}{(n_1+1)^{\alpha+\varepsilon}} + \frac{Cn_2^\alpha}{(n_2+1)^{\alpha+\varepsilon}} + \sum_{n=n_1+1}^{n_2} \left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| |S_n| \\
&\leq \frac{C}{n_1^\varepsilon} + \frac{C}{n_2^\varepsilon} + \sum_{n=n_1+1}^{n_2} \left|\int_n^{n+1} \frac{-s}{x^{s+1}}\, dx\right| |S_n| \\
&\leq \frac{2C}{n_1^\varepsilon} + \sum_{n=n_1+1}^{n_2} \frac{M}{n^{\alpha+\varepsilon+1}} |S_n| \\
&\leq \frac{2C}{n_1^\varepsilon} + CM \sum_{n=n_1+1}^{n_2} \frac{1}{n^{1+\varepsilon}} \\
&\leq \frac{2C}{n_1^\varepsilon} + CM \int_{n_1}^{n_2} \frac{1}{x^{1+\varepsilon}}\, dx \\
&\leq \frac{2C}{n_1^\varepsilon} + \frac{CM}{\varepsilon n_1^\varepsilon}.
\end{aligned}$$

This shows that the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges locally uniformly on $\mathcal{H}_\alpha$.

2 The Riemann Zeta Function

The result of the previous section shows that the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges locally uniformly on $\mathcal{H}_1$ and that the Dirichlet eta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

converges locally uniformly on $\mathcal{H}_0$. On $\mathcal{H}_1$, we have the identity

$$(1-2^{1-s})\zeta(s) = \zeta(s) - \frac{2}{2^s}\zeta(s) = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) - \left(\sum_{n=1}^{\infty} \frac{2}{(2n)^s}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \eta(s).$$

Then the formula

$$\zeta(s) = \frac{1}{1-2^{1-s}}\eta(s)$$

gives an analytic continuation of $\zeta(s)$ to $\mathcal{H}_0$ except with isolated singularities at the points where $1-2^{1-s} = 0$. These are the points $s = 1 - \frac{2\pi i k}{\log 2}$ for $k \in \mathbb{Z}$. Performing the same trick with 3 instead of 2 gives an analytic continuation of $\zeta(s)$ to $\mathcal{H}_0$ except with isolated singularities at the points $s = 1 - \frac{2\pi i j}{\log 3}$ for $j \in \mathbb{Z}$. However, if $1 - \frac{2\pi i k}{\log 2} = 1 - \frac{2\pi i j}{\log 3}$ then $2^j = 3^k$ so $j = k = 0$. This shows that the only possible non-removable singularity of $\zeta(s)$ in $\mathcal{H}_0$ is an isolated singularity at $s = 1$. Now we can expand

$$1 - 2^{1-s} = 1 - e^{(1-s)\log 2} = 1 - \sum_{n=0}^{\infty} \frac{((1-s)\log 2)^n}{n!} = (s-1)\log 2 + [\text{higher order terms}]$$

so

$$\frac{1}{1-2^{1-s}} = \frac{1}{\log 2}(s-1)^{-1} + [\text{higher order terms}].$$

Now $\eta(1) = \log 2$ so

$$\zeta(s) = \frac{1}{1-2^{1-s}}\eta(s) = (s-1)^{-1} + [\text{higher order terms}].$$

This shows that $\zeta(s)$ has a simple pole of residue 1 at $s = 1$.

Theorem 1. *Let $a_1, a_2, \ldots$ be a sequence of complex numbers. Consider the partial sums defined by*

$$S_n = \sum_{k=1}^{n} a_k.$$

Suppose that there is an $\alpha \in (0,1)$ and a $w \in \mathbb{C}$ such that $|S_n - wn| \leq Cn^\alpha$ for sufficiently large n. Then the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

has an analytic continuation to $\mathcal{H}_\alpha$ except for a simple pole of residue w at $s = 1$.

Proof. We can write

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{a_n - w}{n^s} + w\zeta(s)$$

where the sum is analytic on $\mathcal{H}_\alpha$ and where $w\zeta(s)$ has an analytic continuation to $\mathcal{H}_0$ except for a simple pole of residue w at $s = 1$. □

3 Dirichlet L-functions

Lemma 2. *If A is a finite abelian group and if $\chi\colon A \to \mathbb{C}^\times$ is a group homomorphism then*

$$\sum_{a \in A} \chi(a) = \begin{cases} |A| & \chi = 1, \\ 0 & \chi \neq 1. \end{cases}.$$

Proof. If $\chi = 1$ then

$$\sum_{a \in A} \chi(a) = \sum_{a \in A} 1 = |A|.$$

Now suppose that $\chi(b) \neq 1$ for some $b \in A$. Then

$$\sum_{a \in A} \chi(a) = \sum_{a \in A} \chi(ab) = \chi(b) \sum_{a \in A} \chi(a)$$

so

$$(1 - \chi(b)) \sum_{a \in A} \chi(a) = 0.$$

Since $\chi(b) \neq 1$, this shows that $\sum_{a \in A} \chi(a) = 0$. □

Let N be a positive integer and let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a group homomorphism. The associated Dirichlet series is defined by

$$L(s, \chi) = \sum_{\substack{n \geq 1 \\ \gcd(n,N)=1}} \frac{\chi(n + n\mathbb{Z})}{n^s}.$$

Proposition 3. *If $\chi \neq 1$ then $L(s, \chi)$ has an analytic continuation to $\mathcal{H}_0$. If $\chi = 1$ then $L(s, \chi)$ has an analytic continuation to $\mathcal{H}_0$ except for a simple pole of residue $\frac{\varphi(N)}{N}$ at $s = 1$.*

Proof. This follows from Theorem 1 and Lemma 2. □

4 Example From Algebraic Number Theory

Let $\chi\colon (\mathbb{Z}/4\mathbb{Z})^\times \to \mathbb{C}^\times$ be the nontrivial character. Then we have

$$\begin{aligned} 4\zeta(s)L(s, \chi) &= 4\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)\left(\frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots\right) \\ &= \frac{4}{1^s} + \frac{4}{2^s} + \frac{4}{4^s} + \frac{8}{5^s} + \frac{4}{8^s} + \frac{4}{9^s} + \frac{8}{10^s} + \frac{8}{13^s} + \frac{4}{16^s} + \frac{8}{17^s} + \frac{4}{18^s} + \frac{8}{20^s} + \frac{12}{25^s} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ where } a_n = \left|\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = n\}\right|. \end{aligned}$$

Proving this requires some number theory. A more elementary fact is that the partial sums

$$S_n = \sum_{k=1}^{n} a_k = \left|\{(x, y) \in \mathbb{Z}^2 : 1 \leq x^2 + y^2 \leq n\}\right|$$

satisfy $|S_n - \pi n| \leq C\sqrt{n}$ for sufficiently large n. By Theorem 1, $4\zeta(s)L(s, \chi)$ has a simple pole of residue π at $s = 1$. Thus, $L(1, \chi) = \frac{\pi}{4}$ which gives the Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$