Dedekind Schemes

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1 Introduction

The goal of this paper is to generalize the theory of nonsingular curves to include Dedekind domains. In algebraic number theory, one develops ramification theory for extensions of Dedekind domains. There is a similar ramification theory for nonsingular curves. By gluing Dedekind domains together, we will form objects called Dedekind schemes which unify these two theories. This works because the ramification theory developed in algebraic number theory is local in nature, and so easily translates over to Dedekind schemes.

The first section will focus on individual Dedekind schemes in isolation. We will develop the theory of valautions and divisors, and prove the isomorphism between the Picard group and the group of divisors modulo principal divisors. We will also give an example of how this works in the case of an elliptic curve.

In the second section, we will focus on morphisms between Dedekind schemes. In algebraic number theory, the usual process is to start with a Dedekind domain A with field of fractions K, take a finite extension L/K, and consider the integral closure of A in L. We will generalize this construction to Dedekind schemes and finite extensions of their function fields. Finally, we will develop the basic theory of ramification, and demonstrate how it relates to pullback of divisors.

2 The Picard Group of a Dedekind Scheme

2.1 Dedekind Schemes

Definition 1. A **Dedekind Scheme** is a Noetherian integral scheme of dimension 1, all of whose local rings are regular.

There are two key examples of Dedekind schemes to keep in mind. The first is the spectrum of a Dedekind domain. The second is a nonsingular curve over a field.

Lemma 2 (Points of a Dedekind Scheme)**.** *Let X be a Dedekind scheme. Then X has at least 2 points, one of which is the generic point η (which satisfies $\overline{\{\eta\}} = X$), and the rest of which are closed.*

Proof. Since X has dimension 1, X has at least 2 points. Since X is an integral scheme, X has a generic point η satisfying $\overline{\{\eta\}} = X$. Let $x \in X$ and let $y \in \overline{\{x\}}$. Then $\overline{\{y\}} \subseteq \overline{\{x\}} \subseteq \overline{\{\eta\}} = X$ is a chain of irreducible closed subsets of X. Since X has dimension 1, we must have $\overline{\{y\}} = \overline{\{x\}}$ or $\overline{\{x\}} = \overline{\{\eta\}}$. By the uniqueness of generic points, we must have $y = x$ or $x = \eta$. Thus, if $x \neq \eta$ then x is closed. □

The main takeaway from Lemma 2 is that "closed point" is equivalent to "not the generic point η."

Lemma 3 (Open Subsets of a Dedekind Scheme)**.** *Let X be a Dedekind scheme. Then the nonempty open subsets of X are the subsets of the form $X \setminus \{x_1, \ldots, x_k\}$ for finitely many closed points $x_1, \ldots, x_n \in X$.*

Proof. It is clear that if $x_1, \ldots, x_n \in X$ are closed points then $U = X \setminus \{x_1, \ldots, x_k\}$ is an open subset of X. Conversely, let U be a nonempty open subset of X. Note that $\eta \in U$ as otherwise $X \setminus U$ would be a proper closed subset of X containing η. Then every point of $X \setminus U$ is closed. It remains to show that $X \setminus U$ is finite. Since X is quasi-compact, there exists a finite affine open cover $X = \bigcup_{i=1}^n U_i$ of X. Now $U_i \setminus U$ is a closed subset of U_i so $U_i \setminus U = \{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_X(U_i) : I \subseteq \mathfrak{p}\}$ for some ideal I of $\mathcal{O}_X(U_i)$. However, $U_i \setminus U$ consists entirely of closed points, so all $\mathfrak{p} \in U_i \setminus U$ are maximal ideals of $\mathcal{O}_X(U_i)$. In particular, all $\mathfrak{p} \in U_i \setminus U$ are minimal prime ideals over I. Since the ring $\mathcal{O}_X(U_i)$ is Noetherian, $U_i \setminus U$ is finite. Then $X \setminus U = \bigcup_{i=1}^n U_i \setminus U$ is also finite. □

One technical issue that will complicate some of our results is that the generic point η might be open. Luckily, Lemma 3 tells us exactly when this situation occurs.

Corollary 4. *Let X be a Dedekind scheme with generic point η. Then $\{\eta\}$ is open if and only if X is finite.*

This situation could arise when studying curves over finite fields, or when studying the spectrum of a Dedekind domain with only finitely many maximal ideals (although such a Dedekind domain is necessarily a principal ideal domain).

Lemma 5 (The Structure Sheaf of a Dedekind Scheme)**.** *Let X be a Dedekind scheme with generic point η.*

1. *The stalk $\mathcal{O}_{X,\eta}$ is a field.*

2. *For every closed point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a discrete valuation ring with field of fractions $\mathcal{O}_{X,\eta}$.*

3. *Let U be a nonempty affine open subset of X. If $U = \{\eta\}$ then $\mathcal{O}_X(U)$ is the field $\mathcal{O}_{X,\eta}$. If $U \neq \{\eta\}$ then $\mathcal{O}_X(U)$ is a Dedekind domain with field of fractions $\mathcal{O}_{X,\eta}$.*

Proof. The first statement is a general fact about integral schemes. Now let $x \in X$ be a closed point and let U be a nonempty affine open neighborhood of x. Then $\mathcal{O}_X(U)$ is a Noetherian ring of dimension at most 1, whose localization at every prime ideal is a regular local ring. Then $\mathcal{O}_X(U)$ is either a field or a Dedekind domain. Since U contains at least two points (x and η), $\mathcal{O}_X(U)$ must be a Dedekind domain. Here η is the zero ideal of $\mathcal{O}_X(U)$ and x is a maximal ideal of $\mathcal{O}_X(U)$. Then the localization $\mathcal{O}_X(U)_x = \mathcal{O}_{X,x}$ is a discrete valuation ring with field of fractions $\mathcal{O}_X(U)_\eta = \mathcal{O}_{X,\eta}$.

Lastly, let U be a nonempty affine open subset of X. We just showed that if U contains a closed point x then $\mathcal{O}_X(U)$ is a Dedekind domain with field of fractions $\mathcal{O}_{X,\eta}$. If U contains no closed points then $U = \{\eta\}$ so $\mathcal{O}_X(U) = \mathcal{O}_{X,\eta}$. ☐

We now prove a technical lemma regarding invertible sheaves on Dedekind domains.

Lemma 6. *Let X be a Dedekind scheme, let $\mathscr{L}$ be an invertible sheaf on X, let U be a nonempty open subset of X, and let $x \in X$ be any point. Then the diagram*

$$\begin{array}{ccc} \mathscr{L}(U) & \longrightarrow & \mathscr{L}_x \\ & \searrow & \downarrow \\ & & \mathscr{L}_\eta \end{array}$$

commutes, and all maps in the above diagram are injective.

Proof. Let $V \subseteq U$ be an open neighborhood of x on which $\mathscr{L}$ is trivial. Fix an isomorphism $\mathscr{L}_V \cong \mathcal{O}_X|_V$. This gives an isomorphism $\mathscr{L}_\eta \cong \mathcal{O}_{X,\eta}$. If $W \subseteq V$ is an open neighborhood of x then we obtain an isomorphism $\mathscr{L}(W) \cong \mathcal{O}_X(W)$ making the diagram

$$\begin{array}{ccccc} \mathscr{L}(U) & \longrightarrow & \mathscr{L}(W) & \xrightarrow{\sim} & \mathcal{O}_X(W) \\ & \searrow & \downarrow & & \downarrow \\ & & \mathscr{L}_\eta & \xrightarrow{\sim} & \mathcal{O}_{X,\eta} \end{array}$$

commute. Taking the colimit over W gives the commutative diagram

$$\begin{array}{ccccc} \mathscr{L}(U) & \xrightarrow{f_x} & \mathscr{L}_x & \xrightarrow{\sim} & \mathcal{O}_{X,x} \\ & \overset{f_\eta}{\searrow} & \downarrow g & & \downarrow \\ & & \mathscr{L}_\eta & \xrightarrow{\sim} & \mathcal{O}_{X,\eta} \end{array}$$

where the right vertical map $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\eta}$ is injective by Lemma 5. Then g is also injective. In particular, $\ker f_\eta \subseteq \ker f_x$. Since this holds for any $x \in U$, we have

$$\ker f_\eta \subseteq \bigcap_{p \in U} \ker(\mathscr{L}(U) \to \mathscr{L}_p) = \ker\left(\mathscr{L}(U) \to \prod_{p \in U} \mathscr{L}_p\right) = 0$$

which shows that f_η is injective. By the commutivity of the diagram, f_x must also be injective. ☐

In light of Lemma 6, we will view $\mathscr{L}(U) \subseteq \mathscr{L}_x \subseteq \mathscr{L}_\eta$. In the next section, we will obtain a more precise version of this result (Proposition 8) which will allow us to view $\mathscr{L}_x$ as the elements of $\mathscr{L}_\eta$ that are "defined at x" and $\mathscr{L}(U)$ as the elements of $\mathscr{L}_\eta$ that are "defined on all of U".

2.2 Valuations

Let X be a Dedekind scheme and let $x \in X$ be a closed point. By Lemma 5, $\mathcal{O}_{X,x}$ is a discrete valuation ring with field of fractions $\mathcal{O}_{X,\eta}$. There is a valuation $\nu_x^0 \colon \mathcal{O}_{X,\eta} \to \mathbb{Z} \cup \{\infty\}$ coming from $\mathcal{O}_{X,x}$.

Now let $\mathscr{L}$ be an invertible sheaf on X. Consider an open neighborhood U of x on which $\mathscr{L}$ is trivial, and fix an isomorphism $\mathscr{L}|_U \cong \mathcal{O}_X|_U$. The proof of Lemma 6 gives isomorphisms $\mathscr{L}_x \cong \mathcal{O}_{X,x}$ and $\mathscr{L}_\eta \cong \mathcal{O}_{X,\eta}$ making the diagram

$$\begin{array}{ccc} \mathscr{L}_x & \xrightarrow{\sim} & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \mathscr{L}_\eta & \xrightarrow{\sim} & \mathcal{O}_{X,\eta} \end{array}$$

commute. Then there exists a unique function $\nu_x \colon \mathscr{L}_\eta \to \mathbb{Z} \cup \{\infty\}$ making the diagram

$$\begin{array}{ccc} \mathscr{L}_x & \xrightarrow{\sim} & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \mathscr{L}_\eta & \xrightarrow{\sim} & \mathcal{O}_{X,\eta} \\ & \searrow^{\nu_x} & \downarrow{\scriptstyle \nu_x^0} \\ & & \mathbb{Z} \cup \{\infty\} \end{array}$$

commute. This explicit construction of $\nu_x \colon \mathscr{L}_\eta \to \mathbb{Z} \cup \{\infty\}$ from the valuation $\nu_x^0 \colon \mathcal{O}_{X,\eta} \to \mathbb{Z} \cup \{\infty\}$ requires choosing the neighborhood U and the isomorphism $\mathscr{L}|_U \cong \mathcal{O}_X|_U$. We now give an intrinsic characterization of ν_x, showing that ν_x does not depend on these choices.

Lemma 7. *Let π be a uniformizing parameter for the discrete valuation ring $\mathcal{O}_{X,x}$. Then ν_x is given by*

$$\nu_x(s) = \max\{n \in \mathbb{Z} : \pi^{-n} s \in \mathscr{L}_x\}.$$

In particular, ν_x does not depend on the choice of the neighborhood U or the isomorphism $\mathscr{L}|_U \cong \mathcal{O}_X|_U$.

Proof. Let $\varphi \colon \mathscr{L}_\eta \to \mathcal{O}_{X,\eta}$ be the chosen $\mathcal{O}_{X,\eta}$-module isomorphism with $\varphi(\mathscr{L}_x) = \mathcal{O}_{X,x}$. Then

$$\nu_x(s) \geq n \iff \nu_x^0(\varphi(s)) \geq n \iff \pi^{-n}\varphi(s) \in \mathcal{O}_{X,x} \iff \varphi(\pi^{-n}s) \in \mathcal{O}_{X,x} \iff \pi^{-n}s \in \mathscr{L}_x. \qquad \square$$

We should mention that the functions ν_x^0 are really just the functions ν_x associated to $\mathcal{O}_X$.

Proposition 8. *Let X be a Dedekind scheme, let $\mathscr{L}$ be an invertible sheaf on X, let $s, t \in \mathscr{L}_\eta$, and let $r \in \mathcal{O}_{X,\eta}$. Then the functions $\nu_x \colon \mathscr{L}_\eta \to \mathbb{Z} \cup \{\infty\}$ satisfy:*

1. *For each closed point $x \in X$, $\nu_x(s + t) \geq \min(\nu_x(s), \nu_x(t))$ with equality if $\nu_x(s) \neq \nu_x(t)$,*
2. *For each closed point $x \in X$, $\nu_x(rs) = \nu_x^0(r) + \nu_x(s)$,*
3. *For each closed point $x \in X$, $\nu_x(s) = \infty \iff s = 0$,*
4. *For each closed point $x \in X$, $\nu_x(s) \geq 0 \iff s \in \mathscr{L}_x$,*
5. *For each nonempty open subset U of X, $[\nu_x(s) \geq 0$ for all closed points $x \in U] \iff s \in \mathscr{L}(U)$.*

Proof. The first four statements follow from the analogous statements about the valuation ν_x^0. Let U be a nonempty open subset of X. It remains to show that $[s \in \mathscr{L}_x$ for all closed points $x \in U] \iff s \in \mathscr{L}(U)$. The $\Longleftarrow$ direction is clear. Now suppose that $s \in \mathscr{L}_x$ for all closed points $x \in U$. We wish to show that $s \in \mathscr{L}(U)$. By the gluing axiom for sheaves, it suffices to show that $s \in \mathscr{L}(V)$ for every nonempty affine open subscheme $V \subseteq U$ of X on which $\mathscr{L}$ is trivial. Fix an isomorphism $\mathscr{L}|_V \cong \mathcal{O}_X|_V$. By Lemma 5, $\mathcal{O}_X(V)$ is a Dedekind domain with field of fractions $\mathcal{O}_X(V)_\eta = \mathcal{O}_{X,\eta}$. The isomorphism $\mathscr{L}_\eta \cong \mathcal{O}_{X,\eta} = \mathcal{O}_X(V)_\eta$ takes $s \in \mathscr{L}_\eta$ to an element $\widetilde{s} \in \mathcal{O}_X(V)_\eta$ in the field of fractions of $\mathcal{O}_X(V)$. For every closed point $x \in V$, $s \in \mathscr{L}_x$ so $\widetilde{s} \in \mathcal{O}_X(V)_x$. In other words, $\widetilde{s} \in \mathcal{O}_X(V)_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of $\mathcal{O}_X(V)$. Since $\mathcal{O}_X(V)$ is an integral domain, this implies that $\widetilde{s} \in \mathcal{O}_X(V)$. Then $s \in \mathscr{L}(V)$ as desired. $\square$

In light of Proposition 8, it will be helpful to think of the functions ν_x in the following way:

$\nu_x(s)$	means	the order of of vanishing of s at x
$\nu_x(s) \geq 1$	means	s has a zero at x
$\nu_x(s) \leq -1$	means	s has a pole at x
$s \in \mathscr{L}_x$ (or $\nu_x(s) \geq 0$)	means	s is defined at x
$s \in \mathscr{L}(U)$	means	s is defined on all of U

With this in mind, we now show that a nonzero $s \in \mathscr{L}_\eta$ has only finitely many zeros and poles.

Proposition 9. *Let X be a Dedekind scheme, let $\mathscr{L}$ be an invertible sheaf on X, and let $s \in \mathscr{L}_\eta$ be nonzero. Then $\nu_x(s) = 0$ for all but finitely many closed points $x \in X$.*

Proof. Let U be a nonempty affine open subscheme of X on which $\mathscr{L}$ is trivial. If $U = \{\eta\}$ then X is finite by Lemma 3. Thus, we may assume that $U \neq \{\eta\}$. Fix an isomorphism $\mathscr{L}|_U \cong \mathcal{O}_X|_U$. By Lemma 5, $\mathcal{O}_X(U)$ is a Dedekind domain with field of fractions $\mathcal{O}_X(U)_\eta = \mathcal{O}_{X,\eta}$. The isomorphism $\mathscr{L}_\eta \cong \mathcal{O}_{X,\eta} = \mathcal{O}_X(U)_\eta$ takes $s \in \mathscr{L}_\eta$ to a nonzero element $\widetilde{s} \in \mathcal{O}_X(U)_\eta$ in the field of fractions of $\mathcal{O}_X(U)$. Write $\widetilde{s} = a/b$ for nonzero elements $a, b \in \mathcal{O}_X(U)$. Since $\mathcal{O}_X(U)$ is a Dedekind domain, a and b are each contained in only finitely many maximal ideals of $\mathcal{O}_X(U)$. Then $\nu_x(s) = \nu_x^0(\widetilde{s}) = 0$ for all but finitely many closed points $x \in U$. However, $X \setminus U$ is finite by Lemma 3. Thus, $\nu_x(s) = 0$ for all but finitely many closed points $x \in X$. $\square$

One consequence of this is that there is a largest open subset of X on which s is defined.

Corollary 10. *Let X be a Dedekind scheme, let $\mathscr{L}$ be an invertible sheaf on X, and let $s \in \mathscr{L}_\eta$. Then*

$$U_s = \{\text{closed points } x \in X : \nu_x(s) \geq 0\} \cup \{\eta\}$$

is an open subset of X and is the largest open subset of X with $s \in \mathscr{L}(U)$.

Proof. By Proposition 9, U_s is open. By Proposition 8, U_s is the largest open subset of X with $s \in \mathscr{L}(U)$. $\square$

2.3 Divisors

We now turn our attention to divisors on Dedekind schemes.

Definition 11. For a Dedekind scheme X, let $\operatorname{Div} X$ denote the free abelian group on the closed points of X. The elements of $\operatorname{Div} X$ are called **divisors** on X. For an invertible sheaf $\mathscr{L}$ on X and a nonzero $s \in \mathscr{L}_\eta$, we define $\operatorname{div} s = \sum_x \nu_x(s) \cdot [x] \in \operatorname{Div} X$.

Our main result of this first section will be an isomorphism between the Picard group of X and a quotient of $\operatorname{Div} X$. We now prove two key results that will correspond to injectivity and surjectivity of this isomorphism.

Lemma 12 (Injectivity)**.** *Let X be a Dedekind scheme and let $\mathscr{L}$ be an invertible sheaf on X. If there exists a nonzero $s \in \mathscr{L}_\eta$ with $\operatorname{div} s = 0$ then $\mathscr{L} \cong \mathcal{O}_X$.*

Proof. By Proposition 8, there is an $\mathcal{O}_X$-module homomorphism $\varphi \colon \mathcal{O}_X \to \mathscr{L}$ given by $\varphi(r) = rs$. Let $x \in X$ be a closed point and consider the commutative diagram

$$\begin{array}{ccc}
\mathscr{L}_x & \xrightarrow{\sim} & \mathcal{O}_{X,x} \\
\downarrow & & \downarrow \\
\mathscr{L}_\eta & \xrightarrow{\sim} & \mathcal{O}_{X,\eta} \\
& \searrow^{\nu_x} & \downarrow{\scriptstyle \nu_x^0} \\
& & \mathbb{Z} \cup \{\infty\}
\end{array}$$

from the construction of ν_x. Since $\nu_x(s) = 0$, the image of s in $\mathcal{O}_{X,\eta}$ must land in $\mathcal{O}_{X,x}^\times$. In particular, the $\mathcal{O}_{X,x}$-module homomorphism $\varphi_x\colon \mathcal{O}_{X,x} \to \mathscr{L}_x$ is an isomorphism. This holds for all closed points $x \in X$. Let U be a nonempty affine open subscheme of X with $U \neq \{\eta\}$. Then the $\mathcal{O}_X(U)$-module homomorphism $\varphi|_U\colon O_X(U) \to \mathscr{L}(U)$ is an isomorphism since its localization at any maximal ideal is an isomorphism. Localizing at the zero ideal shows that the $\mathcal{O}_{X,\eta}$-module homomorphism $\varphi_\eta\colon \mathcal{O}_{X,\eta} \to \mathscr{L}_\eta$ is an isomorphism. We have shown that φ induces an isomorphism at every stalk. Therefore, φ is an isomorphism. □

Proposition 13 (Surjectivity)**.** *Let X be a Dedekind scheme, let $D = \sum_x n_x \cdot [x] \in \operatorname{Div} X$ be a divisor on X, and let $\mathscr{L}$ be an invertible sheaf on X. Then the data*

$$\mathscr{L}(D)(U) = \begin{cases} \{s \in \mathscr{L}_\eta : \nu_x(s) \geq -n_x \text{ for all closed points } x \in U\} & \text{if } U \neq \varnothing, \\ 0 & \text{if } U = \varnothing \end{cases}$$

defines an invertible sheaf $\mathscr{L}(D)$ on X. There is a $\mathcal{O}_{X,\eta}$-module isomorphism $\varphi\colon \mathscr{L}_\eta \xrightarrow{\sim} \mathscr{L}(D)_\eta$ which satisfies $\operatorname{div}\varphi(s) = \operatorname{div} s + D$ for all nonzero $s \in \mathscr{L}_\eta$.

Proof. We first check that $\mathscr{L}(D)$ is a sheaf. Let U be an open subset of X, let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of U, and fix compatible sections $s_\alpha \in \mathscr{L}(D)(V_\alpha)$. If U is empty then $0 \in \mathscr{L}(D)(U)$ is the unique gluing of the s_α. Now suppose that U is nonempty. If V_α and V_β are two nonempty elements of the cover then considering the generic point η shows that $V_\alpha \cap V_\beta$ is also nonempty. Then the restriction maps $\mathscr{L}(D)(V_\alpha) \to \mathscr{L}(D)(V_\alpha \cap V_\beta)$ and $\mathscr{L}(D)(V_\beta) \to \mathscr{L}(D)(V_\alpha \cap V_\beta)$ are both inclusions. In particular, $s_\alpha = s_\beta$. Since this holds whenever V_α and V_β are nonempty elements of the open cover, we obtain an element $s \in \mathscr{L}_\eta$ such that $s_\alpha = s$ for any index $\alpha \in A$ with V_α nonempty. Also, $s \in \mathscr{L}(D)(U)$ since $\nu_x(s) = \nu_x(s_\alpha) \geq -n_x$ for any closed point $x \in V_\alpha$. Then s is a gluing of the s_α. Uniqueness of s follows from the fact that if V_α is some nonempty element of the cover then the restriction map $\mathscr{L}(D)(U) \to \mathscr{L}(D)(V_\alpha)$ is an inclusion.

We now check that $\mathscr{L}(D)$ is an $\mathcal{O}_X$-module. Fix an open subset U of X. Recall that $\mathscr{L}(D)(U) \subseteq \mathscr{L}_\eta$ and $\mathcal{O}_X(U) \subseteq \mathcal{O}_{X,\eta}$ where $\mathscr{L}_\eta$ is an $\mathcal{O}_{X,\eta}$-module. We just need to check that $rs \in \mathscr{L}(D)(U)$ for every $r \in \mathcal{O}_X(U)$ and $s \in \mathscr{L}(D)(U)$. By Proposition 8,

$$\nu_x(rs) = \nu_x^0(r) + \nu_x(s) \geq \nu_x(s) \geq -n_x$$

for all closed points $x \in U$.

We now check that $\mathscr{L}(D)$ is an invertible sheaf on X. Let $\mathfrak{m} \in X$ be a closed point and let U be a nonempty affine open neighborhood of $\mathfrak{m}$. By Lemma 5, $\mathcal{O}_X(U)$ is a Dedekind domain. Then $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}_X(U)$ and we can choose an element $r \in \mathfrak{m} \setminus \mathfrak{m}^2$. We will now get rid of finitely many "bad" maximal ideals of $\mathcal{O}_X(U)$.

- Since $\mathcal{O}_X(U)$ is a Dedekind domain, there are only finitely many maximal ideals $\mathfrak{m}' \neq \mathfrak{m}$ containing r.
- By Proposition 9, there are only finitely many closed points $\mathfrak{m}' \in U \setminus \{\mathfrak{m}\}$ with $n_{\mathfrak{m}'} \neq 0$.

To get rid of each of these "bad" maximal ideals, localize at an element of $\mathfrak{m}' \setminus \mathfrak{m}$. This has the effect of puncturing U at a finite number of closed points. Then we can assume without loss of generality that:

- $\mathfrak{m}$ is the only maximal ideal of $\mathcal{O}_X(U)$ containing r,
- $n_x = 0$ for all closed points $x \in U \setminus \{\mathfrak{m}\}$.

A more succinct way to say this is that $n_x = \nu_x^0(r^{n_\mathfrak{m}})$ for all closed points $x \in U$. By Proposition 8,

$$\nu_x(r^{n_\mathfrak{m}} s) = \nu_x^0(r^{n_\mathfrak{m}}) + \nu_x(s) = n_x + \nu_x(s)$$

for all $s \in \mathscr{L}_\eta$ and closed points $x \in U$. Then multiplication by $r^{n_\mathfrak{m}}$ gives an isomorphism $\mathscr{L}(D)|_U \cong \mathscr{L}|_U$. By construction, this holds for a collections of U's that covers X. Since $\mathscr{L}$ is an invertible sheaf, so is $\mathscr{L}(D)$. If we let ν'_x denote the valuation on $\mathscr{L}(D)$ then the isomorphism $\cdot r^{n_\mathfrak{m}} : \mathscr{L}(D)|_U \to \mathscr{L}|_U$ gives the identity

$$\nu'_x(s) = \nu_x(r^{n_\mathfrak{m}} s) = n_x + \nu_x(s)$$

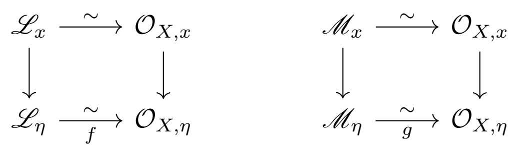
for all $s \in \mathscr{L}_\eta$. Since this holds for all closed points $x \in X$, we have $\operatorname{div}' s = \operatorname{div} s + D$ for all nonzero $s \in \mathscr{L}_\eta$. Here we are implicitly identifying $\mathscr{L}(D)_\eta$ with $\mathscr{L}_\eta$. This is reasonable since $\mathscr{L}(D)(U) = \mathscr{L}(U)$ (with the same $\mathcal{O}_X(U)$-module structure) for any open subset U of X that does not contain the closed points $x \in X$ for which $n_x \neq 0$. Since the collection of such U's is cofinal, taking the colimit gives $\mathscr{L}(D)_\eta = \mathscr{L}_\eta$ (with the same $\mathcal{O}_{X,\eta}$-module stricture). If one wishes to avoid this identification, we could write this as an $\mathcal{O}_{X,\eta}$-module isomorphism $\varphi\colon \mathscr{L}_\eta \xrightarrow{\sim} \mathscr{L}(D)_\eta$ satisfying $\operatorname{div}\varphi(s) = \operatorname{div} s + D$ for all nonzero $s \in \mathscr{L}_\eta$. □

2.4 Tensor Products

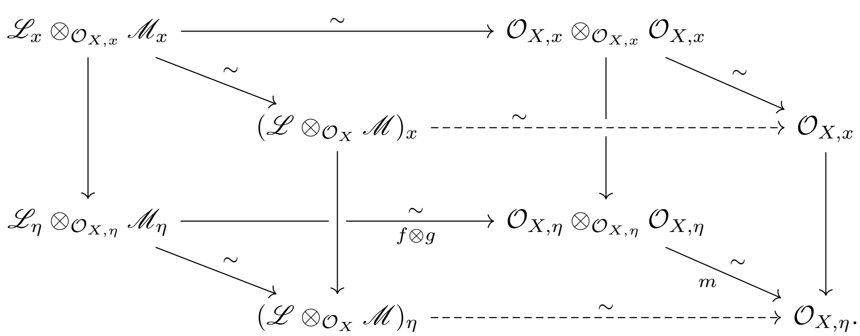
We now show that the group structure on the Picard group is compatible with the group structure on $\operatorname{Div} X$.

Lemma 14. *Let X be a Dedekind scheme, let $\mathscr{L}$ and $\mathscr{M}$ be invertible sheaves on X, and let $s \in \mathscr{L}_\eta$ and $t \in \mathscr{M}_\eta$ be nonzero. Then $s \otimes t \in (\mathscr{L} \otimes \mathscr{M})_\eta$ is nonzero and satisfies* $\operatorname{div}(s \otimes t) = \operatorname{div} s + \operatorname{div} t$.

Proof. Let $x \in X$ be a closed point and fix isomorphisms $f\colon \mathscr{L}_\eta \xrightarrow{\sim} \mathcal{O}_{X,\eta}$ and $g\colon \mathscr{M}_\eta \xrightarrow{\sim} \mathcal{O}_{X,\eta}$ making the diagrams



commute. Then there exist unique isomorphisms $(\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{M})_x \xrightarrow{\sim} \mathcal{O}_{X,x}$ and $(\mathscr{L} \otimes_{\mathcal{O}_X} \mathscr{M})_\eta \xrightarrow{\sim} \mathcal{O}_{X,\eta}$ making the diagram



commute (where the vertical maps are all inclusions). By the construction of ν_x, we have

$$\nu_x(s \otimes t) = \nu_x^0(m((f \otimes g)(s \otimes t))) = \nu_x^0(m(f(s) \otimes g(t))) = \nu_x^0(f(s)g(t)) = \nu_x^0(f(s)) + \nu_x^0(g(t)) = \nu_x(s) + \nu_x(t).$$

Since this holds for all closed points $x \in X$, we have $\operatorname{div}(s \otimes t) = \operatorname{div} s + \operatorname{div} t$. □

Lemma 15. *Let X be a Dedekind scheme, let $\mathscr{L}$ be an invertible sheaf on X, and let $s \in \mathscr{L}_\eta$ be nonzero. Then there exists a unique nonzero element $s^\vee \in \mathscr{L}^\vee$ with $s \otimes s^\vee = 1 \in \mathcal{O}_{X,\eta}$ and* $\operatorname{div} s^\vee = -\operatorname{div} s$.

Proof. There is an $\mathcal{O}_{X,\eta}$-module isomorphism $\mathscr{L}_\eta \otimes (\mathscr{L}^\vee)_\eta \xrightarrow{\sim} \mathcal{O}_{X,\eta}$ where $\mathscr{L}_\eta$ and $(\mathscr{L}^\vee)_\eta$ are $\mathcal{O}_{X,\eta}$-vector spaces of dimension 1. Then there exists a unique nonzero element $s^\vee \in \mathscr{L}^\vee$ with $s \otimes s^\vee = 1 \in \mathcal{O}_{X,\eta}$. By Lemma 14, $\operatorname{div} s^\vee = -\operatorname{div} s$. □

We can now state the main result of this first section. Recall that the Picard group of X is the group of isomorphism classes of invertible sheaves on X under tensor product.

Theorem 16. *Let X be a Dedekind scheme.*

1. *The set* $\operatorname{PrDiv} X = \{\operatorname{div} s : \text{nonzero } s \in \mathcal{O}_{X,\eta}\}$ *is a subgroup of* $\operatorname{Div} X$.

2. *There is an isomorphism* $\varphi\colon \operatorname{Pic} X \xrightarrow{\sim} \operatorname{Div} X/\operatorname{PrDiv} X$ *given by* $\varphi(\mathscr{L}) = \operatorname{div} s + \operatorname{PrDiv} X$ *for any nonzero* $s \in \mathscr{L}_\eta$*. The inverse is given by* $\varphi^{-1}(D + \operatorname{PrDiv} X) = \mathcal{O}_X(D)$*.*

3. *Let* $\mathscr{L}, \mathscr{M}$ *be invertible sheaves on* X*. Then* $\mathscr{M} \cong \mathscr{L}(\operatorname{div} t - \operatorname{div} s)$ *for any nonzero* $s \in \mathscr{L}_\eta$ *and* $t \in \mathscr{M}_\eta$*. In particular,* $\mathscr{L} \cong \mathcal{O}_X(\operatorname{div} s)$ *for any nonzero* $s \in \mathscr{L}_\eta$*.*

Proof. We will prove each part separately.

1. By Lemma 14, there is a group homomorphism $\operatorname{div}\colon \mathcal{O}_{X,\eta}^\times \to \operatorname{Div} X$ with image $\operatorname{PrDiv} X$.

2. Note that if $s, t \in \mathscr{L}_\eta$ are nonzero, then by Lemmas 14 and 15, $s \otimes t^\vee \in \mathcal{O}_{X,\eta}$ is nonzero and satisfies $\operatorname{div}(s \otimes t^\vee) = \operatorname{div} s - \operatorname{div} t$. In particular, $\operatorname{div} s - \operatorname{div} t \in \operatorname{PrDiv} X$ so $\operatorname{div} s + \operatorname{PrDiv} X = \operatorname{div} t + \operatorname{PrDiv} X$. This shows that φ is well-defined. By Lemma 14, φ is a homomorphism. Injectvitity of φ (or rather, $\ker \varphi = 0$) follows from Lemma 12. Surjectivity of φ follows from Proposition 13 which implies that $\varphi(\mathcal{O}_X(D)) = D + \operatorname{PrDiv} X$ for any divisor $D \in \operatorname{Div} X$.

3. Let $D = \operatorname{div} t - \operatorname{div} s$. By Proposition 13, there is a nonzero $u \in \mathscr{L}(D)_\eta$ with $\operatorname{div} u = \operatorname{div} s + D = \operatorname{div} t$. Then $\varphi(\mathscr{M}) = \varphi(\mathscr{L}(D))$ so $\mathscr{M} \cong \mathscr{L}(D)$. Setting $\mathscr{M} = \mathcal{O}_X$ gives the final statement. □

Example 17. Let $X = \mathbb{P}^1_k$. Then $\mathcal{O}_{X,\eta}^\times = k(t)$ and taking the image under $\operatorname{div}\colon \mathcal{O}_{X,\eta}^\times \to \operatorname{Div} X$ shows that $\operatorname{PrDiv} X$ is the subgroup of divisors whose coefficients sum to 0. Equivalently, $\operatorname{PrDiv} X$ is the kernel of the degree homomorphism $\deg\colon \operatorname{Div} X \to \mathbb{Z}$. Then $\operatorname{Pic} X \cong \operatorname{Div} X/\operatorname{PrDiv} X \cong \operatorname{Div} X/\ker \deg \cong \mathbb{Z}$.

Example 18. Let R be a Dedekind domain with field of fractions k, and let $X = \operatorname{Spec} R$. Unique factorization of fraction ideals shows that the group of fractional ideals of R is isomorphic to $\operatorname{Div} X$. Also, $\mathcal{O}_{X,\eta}^\times = k^\times$ and taking the image under $\operatorname{div}\colon \mathcal{O}_{X,\eta}^\times \to \operatorname{Div} X$ shows that $\operatorname{PrDiv} X$ is isomorphic to the subgroup of principal fractional ideals of R. Then the quotient $\operatorname{Div} X/\operatorname{PrDiv} X$ is the ideal class group of X.

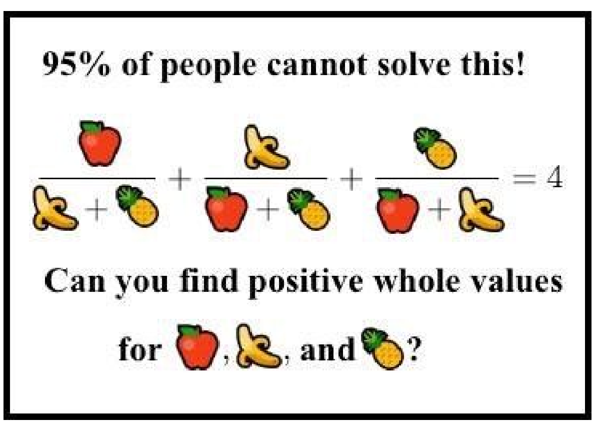
2.5 An Example

We now give an example of how this theory applies to a specific elliptic curve. Hopefully this example will illustrate the beautiful theory of elliptic curves without getting bogged down in machinery.

Motivating Question: Find a positive integer solution to the equation

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} = 4.$$

A while ago, this problem made its rounds on the internet in the form:



For this particular problem, 95% is a vast underestimate. Clearing denominators gives the equation

$$x^3 + y^3 + z^3 + xyz = 3(x+y)(x+z)(y+z)$$

which defines a curve C in $\mathbb{P}^2_{\mathbb{C}}$. There are a few small points on C. For example, there is the point $O = [-1 : 0 : 1]$. The nice feature of O is that the tangent line $y = 6(x+z)$ passes through O with multiplicity 3. This will be relevant later on.

Proposition 19. *The curve C is a Dedekind scheme.*

Proof. We need to check that C is nonsingular. By symmetry, it suffices to check that C is nonsingular on the affine patch $z = 1$. On this affine patch, the equation for C becomes

$$x^3 + y^3 + 1 + xy = 3(x+y)(x+1)(y+1).$$

If C is singular at (x, y) then by the Jacobian criterion,

$$3x^2 + y = 3(2x + y + 1)(y + 1), \quad (1)$$
$$3y^2 + x = 3(2y + x + 1)(x + 1). \quad (2)$$

Subtracting equation (2) from equation (1) gives $(x-y)(6(x+y)-1) = 0$. Thus, $x = y$ or $x + y = \frac{1}{6}$. If $x = y$ then (1) gives $(x, y) = (-\frac{1}{3}, -\frac{1}{3})$ or $(x, y) = (-\frac{3}{2}, -\frac{3}{2})$, neither of which are on C. If $x + y = \frac{1}{6}$ then (1) gives $(x, y) = (\frac{1+\sqrt{95}}{12}, \frac{1-\sqrt{95}}{12})$ or $(x, y) = (\frac{1-\sqrt{95}}{12}, \frac{1+\sqrt{95}}{12})$, neither of which are on C. □

We now describe the group of principal divisors on C. Let $\deg\colon \operatorname{Div} C \to \mathbb{Z}$ be the degree homomorphism given by $\deg(\sum_x n_x \cdot [x]) = \sum_x n_x$. The group of degree zero divisors is denoted by $\operatorname{Div}^0 C = \ker \deg$.

Proposition 20. *The group $\operatorname{PrDiv} C$ of principal divisors on C satisfies:*

1. *If L_1 and L_2 are lines in $\mathbb{P}^2$ then each line L_i intersects C at three points P_i, Q_i, and R_i (counting multiplicity), and $[P_1] + [Q_1] + [R_1] - [P_2] - [Q_2] - [R_2] \in \operatorname{PrDiv} C$.*
2. *$\operatorname{PrDiv} C \subseteq \operatorname{Div}^0 C$.*
3. *If $[P] - [Q] \in \operatorname{PrDiv} C$ then $P = Q$.*

Proof. By Bezout's theorem, each line L_i intersects C at three points. Suppose that the line L_i is defined by the homogeneous equation $a_i x + b_i y + c_i z = 0$. Then the quotient

$$\frac{a_1 x + b_1 y + c_1 z}{a_2 x + b_2 y + c_2 z}$$

is a rational function on C with divisor $[P_1] + [Q_1] + [R_1] - [P_2] - [Q_2] - [R_2]$.

More generally, consider a rational function $\frac{f(x,y,z)}{g(x,y,z)}$ on C, where $f(x, y, z)$ and $g(x, y, z)$ are homogeneous of the same degree d. By Bezout's theorem, $f(x, y, z)$ and $g(x, y, z)$ will each intersect C at $3d$ points (counting multiplicity), so the corresponding principal divisor will have degree 0. Another reference for this result is Corollary II.6.10 of [1].

Now suppose that the rational function $\frac{f(x,y,z)}{g(x,y,z)}$ on C has divisor $[P] - [Q]$. This means that $f(x, y, z)$ and $g(x, y, z)$ intersect C in at least $3d - 1$ shared points (counting multiplicity). Note that $f(x, y, z)$ and $g(x, y, z)$ are nonconstant so $d \geq 1$ and $3d - 1 \geq d + 1$. Since a curve of degree d is defined by $d + 1$ points, f and g must be scalar multiples of each other. Then $\frac{f(x,y,z)}{g(x,y,z)}$ has trivial divisor so $P = Q$. □

Now our special point $O = [-1 : 0 : 1]$ comes in handy. The tangent line $y = 6(x + z)$ passes through O with multiplicity 3. By Proposition 20, if L is a line in $\mathbb{P}^2$ that intersects C at three points P, Q, and R, then $[P] + [Q] + [R] - 3[O] \in \operatorname{PrDiv} C$.

If P and Q are any two points of C, then consider the line L passing through P and Q. We know that L intersects C at another point R, and that $[P]+[Q]+[R]-3[O] \in \operatorname{PrDiv} C$. Then consider the line L' passing through O and R. We know that L' intersects C at another point R', and that $[O]+[R]+[R']-3[O] \in \operatorname{PrDiv} C$. Putting these together gives $[P] + [Q] - [R'] - [O] \in \operatorname{PrDiv} C$. This result is important so we will summarize it in a lemma.

Lemma 21. *Let P and Q be two points of C. Let L be the line passing through P and Q. Let R be the third point of intersection of L with C. Let L' be the line passing through O and R. Let R' be the third point of intersection of L' with C. Then $[P] + [Q] - [R'] - [O] \in \operatorname{PrDiv} C$.*

Recall that if a line L intersects C at P, Q, and R, then $[P] + [Q] + [R] - 3[O] \in \operatorname{PrDiv} C$. We can write this as $([P] - [O]) + ([Q] - [O]) + ([R] - [O]) \in \operatorname{PrDiv} C$. With this in mind, we define a function $f: C \to \operatorname{Div}^0 C/\operatorname{PrDiv} C$ by $f(P) = [P] - [O] + \operatorname{PrDiv} C$.

Theorem 22. *The function $f: C \to \operatorname{Div}^0 C/\operatorname{PrDiv} C$ is a bijection.*

Proof. If $f(P) = f(Q)$ then $[P] - [O] + \operatorname{PrDiv} C = [Q] - [O] + \operatorname{PrDiv} C$ so we have $[P] - [Q] \in \operatorname{PrDiv} C$. By Proposition 20, $P = Q$. This shows that f is injective. Now consider a coset of $\operatorname{Div}^0 C/\operatorname{PrDiv} C$. By adding many copies of $[P] + [Q] + [R] - 3[O]$, we know that this coset contains an element $D = \sum_x n_x \cdot [x]$ such that $n_x \geq 0$ for all $x \neq O$. Among such elements, let D be one that minimizes the quantity $q = \sum_{x \neq O} n_x$. If $q \geq 2$, then subtracting a divisor of the form $[P] + [Q] - [R'] - [O] \in \operatorname{PrDiv} C$ (from Lemma 21) gives a contradiction. Thus, $q \leq 1$. Since $\deg D = 0$, either $D = 0 = [O] - [O]$ or $D = [P] - [O]$. This shows that every coset of $\operatorname{Div}^0 C/\operatorname{PrDiv} C$ is in the image of f. ☐

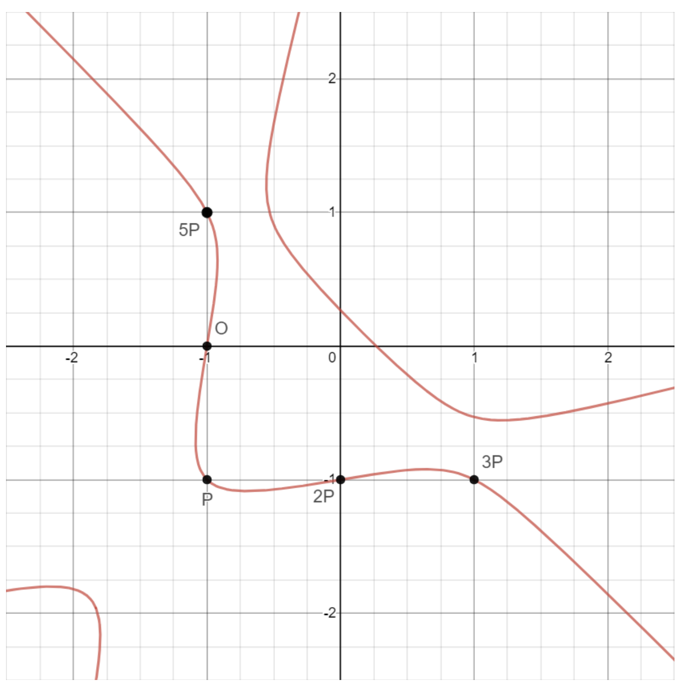
Since f is a bijection, the abelian group structure on $\operatorname{Div}^0 C/\operatorname{PrDiv} C$ induces an abelian group structure on C. For example, in the notation of Lemma 21 we have $f(P) + f(Q) = f(R')$ so $P + Q = R'$. Thus, the group law on C is given by the geometric process described in Lemma 21.

We now use the group law on C to solve our motivating question. The idea is to take small rational points that are easy to find, and add them together to obtain a rational point with positive coordinates.

Let's start with the point $P = [-1 : -1 : 1]$. To compute $2P$, take the tangent line $x + y + 2z = 0$ through P. Its third point of intersection with C is the point $-2P = [-1 : 1 : 0]$. The line through $O = [-1 : 0 : 1]$ and $-2P$ is the line $x + y + z = 0$. Its third point of intersection with C is the point $2P = [0 : -1 : 1]$. Repeating this process constructs the rational points

$$
\begin{aligned}
O &= [-1 : 0 : 1], & P &= [-1 : -1 : 1], & 2P &= [0 : -1 : 1], \\
3P &= [1 : -1 : 1] & 4P &= [-1 : 1 : 0], & 5P &= [-1 : 1 : 1].
\end{aligned}
$$

The graph below depicts these points in the affine patch $z = 1$.



The good news is that we've found a subgroup of C of order 6. The bad news is that we're stuck until we find a new rational point. A small search finds the rational point $Q = [4 : -1 : 11]$. Then taking multiples of Q give the rational points

$$1Q = \begin{bmatrix} 4 \\ -1 \\ 11 \end{bmatrix}, \quad 2Q = \begin{bmatrix} -5165 \\ -9499 \\ 8784 \end{bmatrix}, \quad 3Q = \begin{bmatrix} -375326521 \\ 883659076 \\ 679733219 \end{bmatrix}, \quad 4Q = \begin{bmatrix} 6696085890501216 \\ -6531563383962071 \\ 6334630576523495 \end{bmatrix}$$

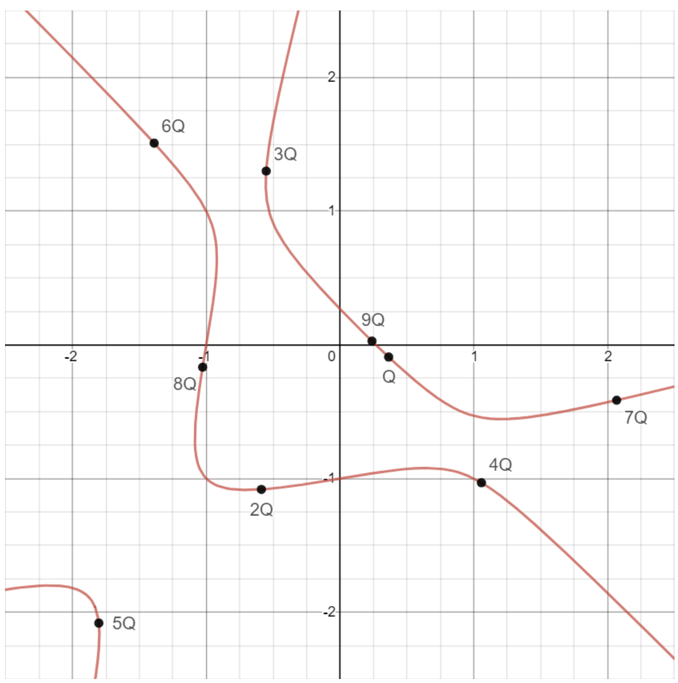
$$5Q = \begin{bmatrix} -5048384306267455380784631 \\ -5824662475191962424632819 \\ 2798662276711559924688956 \end{bmatrix}, \quad 6Q = \begin{bmatrix} -3998662586244387372324936462444383709 \\ 4340214040910911407820002345916l8320 \\ 2876630488972245543374469183444054 29 \end{bmatrix}$$

$$7Q = \begin{bmatrix} 33869282463293272597638491845101850314062113248 04 \\ -6782669709301339235789161616483503982063541013 81 \\ 1637627722378544613543242758851617912968156867151 \end{bmatrix}$$

$$8Q = \begin{bmatrix} -211076064923132585504708897456046866753261616439752014262210446 5 \\ -34325830325463534321117548458857243057528993892765697220156379 1 \\ 20542177039801989407659936215672608347918166641490062173060677 76 \end{bmatrix}$$

$$9Q = \begin{bmatrix} 3687513179412999982719781156522547482549297996897197099628313747163722463405557 9 \\ 437361267792869725786125260237139015281653755816161361862143799337842346777203 6 \\ 15447680210874616644195131501991983748566432566956543170002663489825320203527799 9 \end{bmatrix}$$

The graph below depicts these points in the affine patch $z = 1$.



Once we reach $9Q$, we obtain a rational point with positive coordinates. In fact, this is the smallest positive integer solution to our equation!

$$\begin{aligned} x &= 36875131794129999827197811565225474825492979968971970996283137471637224634055579, \\ y &= 43736126779286972578612526023713901528165375581616136186214379933784234677720 36, \\ z &= 154476802108746166441951315019919837485664325669565431700026634898253202035277999. \end{aligned}$$

We should mention that the group $C(\mathbb{Q})$ of rational points of C is isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}$, generated by P and Q. In other words, $C(\mathbb{Q})$ has rank 1 and torsion subgroup $\mathbb{Z}/6\mathbb{Z}$.

We conclude by determining $\operatorname{Pic} C$. There is a short exact sequence of abelian groups

$$0 \longrightarrow \operatorname{Div}^0 C \longrightarrow \operatorname{Div} C \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

Taking the quotient by $\operatorname{PrDiv} C$ gives a short exact sequence

$$0 \longrightarrow \frac{\operatorname{Div}^0 C}{\operatorname{PrDiv} C} \longrightarrow \frac{\operatorname{Div} C}{\operatorname{PrDiv} C} \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

By Theorems 16 and 22, we can rewrite this short exact sequence as

$$0 \longrightarrow C \longrightarrow \operatorname{Pic} C \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

Abstractly, this sequence splits (noncanonically) so we obtain $\operatorname{Pic} C \cong C \oplus \mathbb{Z}$.

3 Ramification Theory

3.1 Finite Morphisms

We will restrict our attention to nonconstant finite morphisms of Dedekind schemes. These morphisms are the natural generalization of the "AKLB" setup for Dedekind domains (see Lemma 24).

Lemma 23. *Let $f\colon Y \to X$ be a nonconstant finite morphism of Dedekind schemes.*

- $f(\eta_Y) = \eta_X$.
- *The field homomorphism $\mathcal{O}_{X,\eta_X} \to \mathcal{O}_{Y,\eta_Y}$ is injective and the field extension $\mathcal{O}_{Y,\eta_Y}/\mathcal{O}_{X,\eta_X}$ is finite.*
- *If U is an affine open subscheme of X then $V = f^{-1}(U)$ is an affine open subscheme of Y, and $\mathcal{O}_Y(V)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{O}_{Y,\eta_Y}$.*

Proof. We know that $f(Y) = f(\overline{\{\eta_Y\}}) \subseteq \overline{\{f(\eta_Y)\}}$. Since f is nonconstant, $f(\eta_Y)$ cannot be a closed point of X. Then we must have $f(\eta_Y) = \eta_X$. Let U be a nonempty affine open subscheme of X and let $V = f^{-1}(U)$. Since f is finite, we know that V is an affine open subscheme of Y, and that the ring homomorphism $\varphi\colon \mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ makes $\mathcal{O}_Y(V)$ into a finite $\mathcal{O}_X(U)$-module. Since $f(\eta_Y) = \eta_X$, we have $\ker\varphi = \varphi^{-1}((0)) = (0)$ which shows that φ is injective. Passing to the fields of fractions shows that the field homomorphism $\mathcal{O}_{X,\eta_X} \to \mathcal{O}_{Y,\eta_Y}$ is injective. Since $\mathcal{O}_Y(V)$ is a finite $\mathcal{O}_X(U)$-module, passing to the fields of fractions shows that $\mathcal{O}_{Y,\eta_Y}/\mathcal{O}_{X,\eta_X}$ is finite. By Lemma 5, $\mathcal{O}_Y(V)$ is integrally closed in its field of fractions $\mathcal{O}_{Y,\eta_Y}$. Since $\mathcal{O}_Y(V)$ is integral over $\mathcal{O}_X(U)$, $\mathcal{O}_Y(V)$ must be the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{O}_{Y,\eta_Y}$. □

Lemma 23 shows that if $Y \to X$ is a nonconstant finite morphism of Dedekind schemes, then the rings $\mathcal{O}_Y(V)$ are determined by the finite extension $\mathcal{O}_{Y,\eta_Y}/\mathcal{O}_{X,\eta_X}$. We now show how to reconstruct the Dedekind scheme Y from just the Dedekind scheme X and the finite extension $\mathcal{O}_{Y,\eta_Y}/\mathcal{O}_{X,\eta_X}$. We will need the following result from number theory. Many books only state this result for L/K separable, but it is true in general (see the Krull-Akizuki theorem).

Lemma 24. *Let A be a Dedekind domain with field of fractions K, let L be a finite extension of K, and let B be the integral closure of A in L. Then B is a Dedekind domain with field of fractions L.*

Proposition 25. *Let X be a Dedekind scheme with function field $K = \mathcal{O}_{X,\eta_X}$. Let L/K be a finite extension. Then there exists a unique nonconstant finite morphism of Dedekind schemes $Y \to X$ such that the associated field homomorphism $\mathcal{O}_{X,\eta_X} \to \mathcal{O}_{Y,\eta_Y}$ is the inclusion $K \hookrightarrow L$.*

Proof. Let $\{U_i\}_{i=1}^n$ be a finite affine open cover of X. Let $A_i = \mathcal{O}_X(U_i)$, let B_i be the integral closure of A_i in L, and let $V_i = \operatorname{Spec} B_i$. The inclusion $A_i \hookrightarrow B_i$ induces a morphism of schemes $\varphi_i : V_i \to U_i$. For indices $i \neq j$, set $U_{ij} = U_i \cap U_j$ and $V_{ij} = \varphi_i^{-1}(U_{ij})$. Since U_{ij} is an intersection of the affine open subschemes U_i and U_j of X, we can cover U_{ij} with open subsets W_{ijk} which are simultaneously basic affine open subsets of both U_i and U_j. If $W_{ijk} = \operatorname{Spec}((A_i)_f)$ then W_{ijk} consists of the primes of A_i not containing f so $\varphi_i^{-1}(W_{ijk})$ consists of the prime ideals of B_i not containing f. Then $\varphi_i^{-1}(W_{ijk}) = \operatorname{Spec}((B_i)_f)$. Since integral closure commutes with localization, $(B_i)_f$ is the integral closure of $(A_i)_f$ in L. In other words, $\varphi_i^{-1}(W_{ijk})$ is the spectrum of the integral closure in L of the global sections of W_{ijk}. The same holds for $\varphi_j^{-1}(W_{ijk})$. Thus, $\varphi_i^{-1}(W_{ijk}) = \varphi_j^{-1}(W_{ijk})$. Taking the union over k gives $V_{ij} = V_{ji}$. We also have

$$V_{ij} \cap V_{ik} = \varphi^{-1}(U_{ij}) \cap \varphi^{-1}(U_{ik}) = \varphi^{-1}(U_{ij} \cap U_{ik}) = \varphi^{-1}(U_{ji} \cap U_{jk}) = \varphi^{-1}(U_{ji}) \cap \varphi^{-1}(U_{jk}) = V_{ji} \cap V_{jk}.$$

Then we can glue the schemes V_{ij} to obtain a scheme Y. By construction, there is a nonconstant morphism $Y \to X$. Since each B_i is finite over A_i, this morphism is finite. By Lemma 5, each A_i is a field or Dedekind domain with field of fractions K. By Lemma 24, each B_i is a field or Dedekind domain with field of fractions L. Then Y is a Dedekind scheme with $\mathcal{O}_{Y,\eta_Y} = L$, and the field homomorphism $\mathcal{O}_{X,\eta_X} \to \mathcal{O}_{Y,\eta_Y}$ is the inclusion $K \hookrightarrow L$. It remains to show uniqueness. By Lemma 23, any such morphism $Y \to X$ arises via this construction. Then uniqueness of Y follows from uniqueness of gluing schemes. □

We have demonstrated how to go back and forth between finite morphisms of Dedekind schemes $Y \to X$ and finite extensions $\mathcal{O}_{X,\eta_X}$. The reader who knows some category theory may realize that we have actually given an equivalence (or rather, a duality) of categories. The details are left to the reader.

Corollary 26. *Let X be a Dedekind scheme. Then there is a duality of categories between nonconstant finite morphisms of Dedekind schemes $Y \to X$ and finite extensions of $\mathcal{O}_{X,\eta_X}$.*

3.2 Divisors

We now define the ramification and inertia degrees of nonconstant finite morphisms of Dedekind schemes.

Definition 27. Let $Y \to X$ be nonconstant finite morphism of Dedekind schemes. Let $y \in Y$ be a closed point and let $x = f(y)$. Let π be a uniformizing parameter for the discrete valuation ring $\mathcal{O}_{X,x}$. We define $e_{y/x} = \nu_y(\pi)$. We define $f_{y/x} = [\kappa_y : \kappa_x]$ where κ_x and κ_y are the residue fields of x and y respectively.

Note that for a nonsingular curve over an algebraically closed field, we always have $f_{y/x} = 1$. The following lemma will show that ramification degree is well-defined.

Lemma 28. *Let $Y \to X$ be nonconstant finite morphism of Dedekind schemes. Let $y \in Y$ be a closed point and let $x = f(y)$. Then the valuation $\nu_y(\pi)$ does not depend on the choice of uniformizing parameter π for the discrete valuation ring $\mathcal{O}_{X,x}$.*

Proof. If π' is another uniformizing parameter for the discrete valuation ring $\mathcal{O}_{X,x}$, then π and π' differ by a unit of $\mathcal{O}_{X,x}$. Passing to $\mathcal{O}_{Y,y}$ shows that π and π' differ by a unit of $\mathcal{O}_{Y,y}$, so $\nu_y(\pi) = \nu_y(\pi')$. □

We can use our ramification indices to define a pullback homomorphism on divisors.

Definition 29. Let $f : Y \to X$ be a nonconstant finite morphism of Dedekind schemes. We define the pullback homomorphism $f^* : \operatorname{Div} X \to \operatorname{Div} Y$ by $f^*([x]) = \sum_{f(y)=x} e_{y/x} \cdot [y]$.

The pullback homomorphism $f^* : \operatorname{Div} X \to \operatorname{Div} Y$ is compatible with pullback of invertible sheaves.

Lemma 30. *Let $f : Y \to X$ be a nonconstant finite morphism of Dedekind schemes. Let $\mathscr{L}$ be an invertible sheaf on X and let $s \in \mathscr{L}_{\eta_X}$ be nonzero. Then the pullback $f^*s \in (f^*\mathscr{L})_{\eta_Y}$ satisfies* $\operatorname{div} f^*s = f^* \operatorname{div} s$.

Proof. Let $y \in Y$ be a closed point and let $x = f(y)$. Let π_x and π_y be uniformizing parameters for the discrete valuation ring $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ respectively. Then $f^*\pi_x$ and $\pi_y^{e_{y/x}}$ differ by a unit of $\mathcal{O}_{Y,y}$. Under a local trivialization of $\mathscr{L}$, s differs from $\pi_x^{\nu_x(s)}$ by a unit of $\mathcal{O}_{X,x}$. Then f^*s differs from $\pi_y^{e_{y/x}\nu_x(s)}$ by a unit of $\mathcal{O}_{Y,y}$. This shows that $\nu_y(f^*s) = e_{y/x}\nu_x(s)$. If we fix x and sum over y with $f(y) = x$ then we obtain

$$\sum_{f(y)=x} \nu_y(f^*s) \cdot [y] = \sum_{f(y)=x} e_{y/x}\nu_x(s) \cdot [y] = \nu_x(s) \sum_{f(y)=x} e_{y/x} \cdot [y] = \nu_x(s) f^*([x]).$$

Finally, summing over x gives

$$\operatorname{div} f^*s = \sum_y \nu_y(f^*s) \cdot [y] = \sum_x \sum_{f(y)=x} \nu_y(f^*s) \cdot [y] = \sum_x \nu_x(s) f^*([x]) = f^* \left(\sum_x \nu_x(s) \cdot [x] \right) = f^* \operatorname{div} s. \quad \square$$

Note that Lemma 30 completely determines the pullback homomorphism $f^* \colon \operatorname{Div} X \to \operatorname{Div} Y$ since every element $D \in \operatorname{Div} X$ is of the form $\operatorname{div} s$ for some nonzero $s \in \mathcal{O}_X(D)_{\eta_X}$.

We can also view the compatibility result of Lemma 30 as stating the commutativity of a certain diagram.

Proposition 31. *Let $Y \to X$ be a nonconstant finite morphism of Dedekind schemes. Then the diagram*

$$\begin{array}{ccc} \mathcal{O}_{X,\eta_X}^\times & \xrightarrow{f^*} & \mathcal{O}_{Y,\eta_Y}^\times \\ \downarrow & & \downarrow \\ \operatorname{Div} X & \xrightarrow{f^*} & \operatorname{Div} Y \\ \downarrow & & \downarrow \\ \operatorname{Pic} X & \xrightarrow{f^*} & \operatorname{Pic} Y \end{array}$$

commutes.

Proof. The top square commutes by Lemma 30 with $\mathscr{L} = \mathcal{O}_X$. For the bottom square, let $D \in \operatorname{Div} X$. By Proposition 13, there is a nonzero $s \in \mathcal{O}_X(D)_{\eta_X}$ with $\operatorname{div} s = D$. By Lemma 30, $\operatorname{div} f^*s = f^* \operatorname{div} s = f^*D$. By Theorem 16, $f^*\mathcal{O}_X(D) \cong \mathcal{O}_Y(f^*D)$ which shows that the bottom square commutes. $\square$

We remark that if $\operatorname{Pic} X$ is torsion (as in the case of the ring of integers of a number field) then commutivity of the top square completely determines the pullback homomorphism $f^* \colon \operatorname{Div} X \to \operatorname{Div} Y$.

3.3 Ramification Theory

Since ramification is local, we can take ramification theory for Dedekind domains and translate it over to Dedekind schemes.

Theorem 32 (Ramification Theory for Dedekind Schemes)**.** *Let $Y \to X$ be a nonconstant finite morphism of Dedekind schemes. Let $K = \mathcal{O}_{X,\eta_X}$ and let $L = \mathcal{O}_{Y,\eta_Y}$.*

1. *Let $x \in X$ be a closed point and let $y_1, \dots, y_g \in Y$ be the closed points of Y lying over x. Then $\sum_{i=1}^g e_{y_i/x} f_{y_i/x} = [L : K]$. In particular, $g \geq 1$ so f is surjective.*

2. *There are only finitely many closed points $y \in Y$ with $e_{y/f(y)} > 1$.*

3. *If L/K is Galois then* $\operatorname{Gal}(L/K)$ *acts transitively on the closed points $y_i \in Y$ lying over a given closed point $x \in X$, so $e_{y_1/x} = \cdots = e_{y_g/x} = e$ and $f_{y_1/x} = \cdots = f_{y_g/x} = f$ where $efg = [L : K]$.*

Proof. By ramification theory for Dedekind domains, these hold when X is affine. To obtain the general result, take a finite affine open cover of X. $\square$

Corollary 33. *Let $C \to D$ be a nonconstant finite morphism of nonsingular curves over an algebraically closed field k. Then* $\deg f^*D = \deg f \cdot \deg D$.

Proof. As mentioned earlier, $f_{y/x} = 1$ since the residue fields κ_x and κ_y are finite extensions of k. Then

$$\deg f^*([x]) = \deg \sum_{f(y)=x} e_{y/x} \cdot [y] = \sum_{f(y)=x} e_{y/x} f_{y/x} = [L : K] = \deg f$$

and the result follows from linearity. □

Corollary 34. *Let C be a nonsingular curve over an algebraically closed field k. Then* $\operatorname{PrDiv} C \subseteq \operatorname{Div}^0 C$.

Proof. Let $s \in \mathcal{O}_{C,\eta}^{\times}$. If s is constant then $\operatorname{div} s = 0$. If s is nonconstant then s gives a nonconstant finite morphism $\varphi \colon C \to \mathbb{P}^1$ and

$$\deg \operatorname{div} s = \deg \varphi^*([0] - [\infty]) = \deg([0] - [\infty]) \cdot \deg \varphi = 0 \cdot \deg \varphi = 0.$$

In either case, $\operatorname{div} s \in \operatorname{Div}^0 C$. □

This gives another proof of part 2 of Proposition 20.

References

[1] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52, MR 0463157.