Restricted Stacks as Functions

Katalin Berlow

Carnegie Mellon University

kberlow@andrew.cmu.edu

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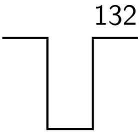

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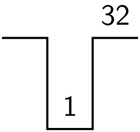

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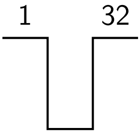

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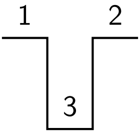

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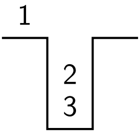

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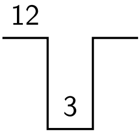

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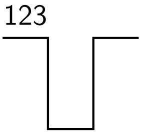

Pattern Avoidance

Order Isomorphic

Let σ and τ be length n words. We say that σ is order isomorphic to τ, in symbols $\sigma \cong \tau$, if and only if for all $i, j \leq n$ we have

$$\sigma(i) < \sigma(j) \text{ if and only if } \tau(i) < \tau(j)$$

$$\sigma(i) > \sigma(j) \text{ if and only if } \tau(i) > \tau(j).$$

For example $425 \cong 213$ but $312 \not\cong 213$.

Pattern Avoidance

Pattern Avoidance

Let σ be a word of length n and τ a permutation of length m. We say that σ *contains* τ if and only if σ has a (not necessarily contiguous) subsequence which is order-isomorphic to τ.

Otherwise we say that σ is τ-avoiding. We say that σ is T-avoiding, for a set T of permutations, if and only if π avoids every element of T.

For example 34521 contains 231 while 14325 does not.
For any length 3 permutation σ, the subset of S_n which avoids σ is of size C_n.

The T-Avoiding Stack

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Let T be a set of permutations. The map s_T sorts permutations (or words) according to the following algorithm: If adding the next element of the input to the stack keeps the stack T-avoiding, then move that element onto the stack. Otherwise, move the top element off the stack and append it to the output.

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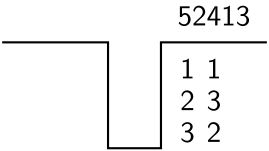

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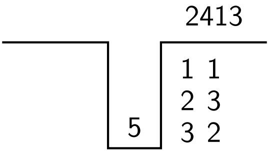

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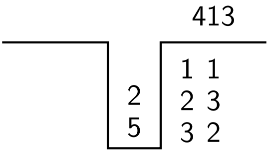

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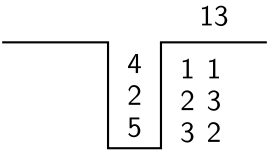

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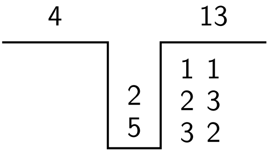

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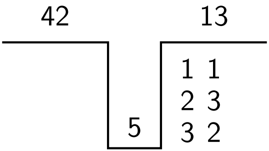

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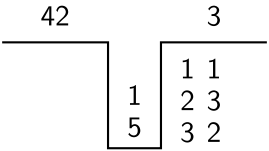

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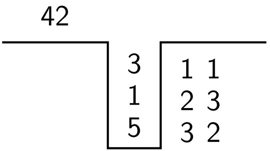

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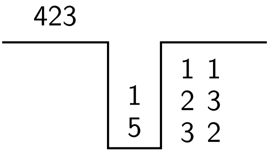

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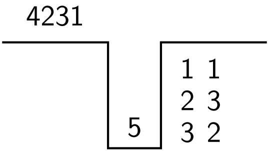

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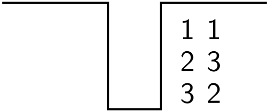

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Let T be a set of permutations. We say that T is *reduced* if and only if there are no $\sigma, \tau \in T$ such that σ contains τ.

For distinct sets of reduced permutations T and R we have that s_T and s_R are distinct. It is also straightforward that for any set of permutations T' there exists a reduced set T such that $s_{T'} = s_T$. Thus, it is sufficient to only consider s_T for reduced sets T.

When is s_T Bijective?

For any length k permutation σ, let $\sigma(i)$ denote the ith entry of σ. We also let $\hat{\sigma}$ denote $\sigma(2)\sigma(1)\sigma(3)\cdots\sigma(k)$.

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Theorem 1 (B.)

Let T be a reduced set of permutations. The map s_T is bijective if and only if for every $\sigma \in T$, we also have $\hat{\sigma} \in T$.

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Theorem 2 (B.)

Let T be a reduced set of permutations. The map s_T is bijective if and only if for every $\sigma \in T$, we also have $\hat{\sigma} \in T$.

Let r be the map which reverses permutations. When s_T is bijective, its inverse is $r \circ s_T \circ r$. inverse is $r \circ s_T \circ r$.

A Corollary

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$(123, 132)$	1 2 5 14	$(123, 321)$	1 2 4 7	$(213, 231)$	1 2 5 16
$(123, 213)$	1 2 5 14	$(132, 213)$	1 2 5 15	$(213, 321)$	1 2 4 12
$(123, 231)$	1 2 6 21	$(132, 312)$	1 2 5 14	$(231, 312)$	1 2 6 23
$(123, 231)$	1 2 6 21	$(132, 321)$	1 2 4 10	$(231, 321)$	1 2 5 14
$(123, 312)$	1 2 5 15	$(213, 231)$	1 2 6 23	$(312, 321)$	1 2 4 10

Preimages

Theorem 4 (B.)

If T is a set of permutations, all of length at least k, then every permutation of length n has at most C_{n-k+2} preimages under the map s_T.

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Theorem 5 (B.)

Let σ be a length k permutation. If $\sigma(1)$ and $\sigma(2)$ are consecutive numbers, then for every $n \geq k$, there exists a permutation $\pi \in S_n$ such that $|s_T^{-1}(\pi)| = C_{n-k+2}$. If $\sigma(0)$ and $\sigma(1)$ are not consecutive, then for every $n > k$ there are no $\pi \in S_n$ such that $|s_T^{-1}(\pi)| = C_{n-k+2}$.

A Specific Case

Corollary 6 (B.)

Let $\pi \in S_n$ be a permutation. Then $s_{213}(\pi) = id_n$ if and only if $\pi = n\rho$ for 231-avoiding $\rho \in S_{n-1}$.

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Theorem 7

The maximal number of preimages for any permutation of length n under the map $s_{213,231}$ is C_{n-1}. This maximum is attained only by id_n and id_n^r.

Periodicity

Periodic Point

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For a given set of permutations T, what are the periodic points of s_T

Periodicity of $S_{\{123,132\}}$

Half-Decreasing

Let π be a permutation of length n. We say that π is *half-decreasing* if the subsequence

$$\pi(n-1)\pi(n-3)\cdots\pi(2) \text{ for odd } n$$

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is the identity of length $\lfloor \frac{n}{2} \rfloor$. (Being order isomorphic to the identity is not sufficient.) We will refer to this subsequence as its *decreasing half*.

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For example, 647352819 and 75382614 are half-decreasing,

$s_{\{123,132\}}$ Acting on Half-Decreasing Permutations

Lemma 8

If π is a half-decreasing permutation of length n then the map $s_{\{123,132\}}$ acts on it as follows:

$$s_{\{123,132\}}(\pi) = \pi(3)\pi(2)\pi(5)\cdots\pi(n)\pi(n-1)\pi(1) \textit{ for odd } n$$

$$s_{\{123,132\}}(\pi) = \pi(2)\pi(4)\pi(3)\pi(6)\cdots\pi(n)\pi(n-1)\pi(1) \textit{ for even } n.$$

In other words, the decreasing half is fixed under $s_{\{123,132\}}$ and the remaining elements shift cyclically to the left.

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So,

$$s^0_{\{123,132\}}(647352819) = 647352819$$

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$$s^1_{\{123,132\}}(647352819) = 745382916$$

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$$s^2_{\{123,132\}}(647352819) = 548392617$$

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$$s^3_{\{123,132\}}(647352819) = 849362715$$

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$$s^5_{\{123,132\}}(647352819) = 946372518$$

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So,

$$s^6_{\{123,132\}}(6\color{red}{47}\color{black}{3}\color{red}{5}\color{black}{2}\color{red}{81}\color{black}{9}) = 6\color{red}{47}\color{black}{3}\color{red}{5}\color{black}{2}\color{red}{81}\color{black}{9}$$

Periodic Points of $s_{\{123,132\}}$

Lemma 9

Let π be a permutation. Then $s^m_{\{123,132\}}$ is half-decreasing for some $m \in$.

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Theorem 10

The periodic points of $s_{\{123,132\}}$ are exactly the half-decreasing permutations.

Further Research

Conjecture

The only periodic points of $s_{132,213}$ and $s_{231,213}$ are the identity and its reverse.

Other Questions:

- Given a set of permutations T, can one find a classification based on T of the maximum number of preimages under the map s_T?
- For a given set of permutations T what is the size of the image of s_T?

Acknowledgements

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The End