Separating complexity classes of LCL problems on grids

Katalin Berlow, Anton Bernshteyn, Clark Lyons, Felix Weilacher

SEALS 2025

Table of Contents

Table of Contents

Schreier Graphs

Definition:

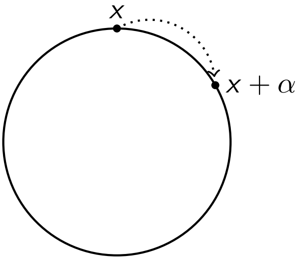
Let $\Gamma = \langle S \rangle \curvearrowright X$ be a Borel action of a finitely generated group on a standard Borel space. The **Schreier graph** of this action is the Borel graph $G \subseteq X \times X$ where we have $(x, y) \in G$ iff $\exists s \in S\ s \cdot x = y$.

Schreier Graphs

Definition:

Let $\Gamma = \langle S \rangle \curvearrowright X$ be a Borel action of a finitely generated group on a standard Borel space. The **Schreier graph** of this action is the Borel graph $G \subseteq X \times X$ where we have $(x, y) \in G$ iff $\exists s \in S \; s \cdot x = y$.

Example 1: Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Take $\mathbb{Z} = \langle a \rangle \curvearrowright \mathbb{R}/\mathbb{Z}$ by $a \cdot x = x + \alpha$.

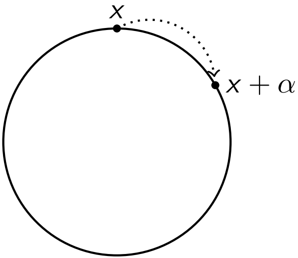


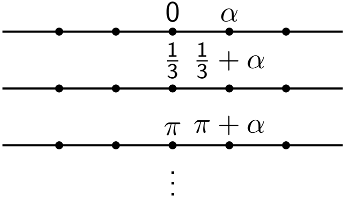
Schreier Graphs

Definition:

Let $\Gamma = \langle S \rangle \curvearrowright X$ be a Borel action of a finitely generated group on a standard Borel space. The **Schreier graph** of this action is the Borel graph $G \subseteq X \times X$ where we have $(x, y) \in G$ iff $\exists s \in S \; s \cdot x = y$.

Example 1: Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Take $\mathbb{Z} = \langle a \rangle \curvearrowright \mathbb{R}/\mathbb{Z}$ by $a \cdot x = x + \alpha$.



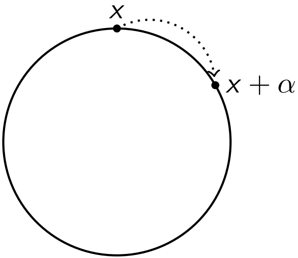


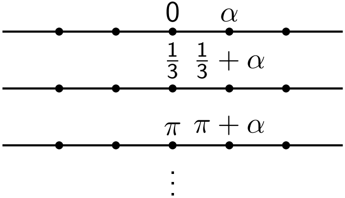
Schreier Graphs

Definition:

Let $\Gamma = \langle S \rangle \curvearrowright X$ be a Borel action of a finitely generated group on a standard Borel space. The **Schreier graph** of this action is the Borel graph $G \subseteq X \times X$ where we have $(x, y) \in G$ iff $\exists s \in S \; s \cdot x = y$.

Example 1: Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Take $\mathbb{Z} = \langle a \rangle \curvearrowright \mathbb{R}/\mathbb{Z}$ by $a \cdot x = x + \alpha$.





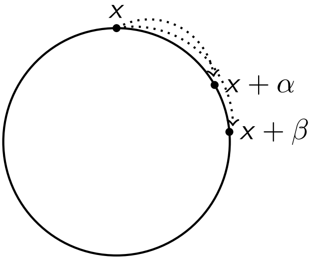
Each connected component will look like a copy of the Cayley graph of $\mathbb{Z}$ because the action is free.

Schreier Graphs

Definition:

Let $\Gamma = \langle S \rangle \curvearrowright X$ be a Borel action of a finitely generated group on a standard Borel space. The **Schreier graph** of this action is the Borel graph $G \subseteq X \times X$ where we have $(x, y) \in G$ iff $\exists s \in S \; s \cdot x = y$.

Example 2: Fix independent $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$. Take $\mathbb{Z}^2 = \langle a, b \rangle \curvearrowright \mathbb{R}/\mathbb{Z}$ by $a \cdot x = x + \alpha$ and $b \cdot x = x + \beta$.

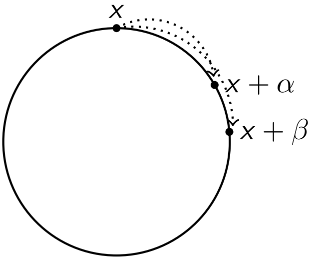


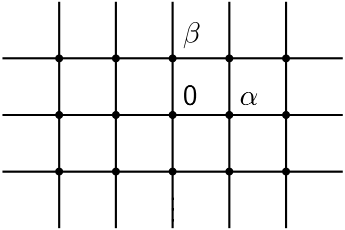
Schreier Graphs

Definition:

Let $\Gamma = \langle S \rangle \curvearrowright X$ be a Borel action of a finitely generated group on a standard Borel space. The **Schreier graph** of this action is the Borel graph $G \subseteq X \times X$ where we have $(x, y) \in G$ iff $\exists s \in S \; s \cdot x = y$.

Example 2: Fix independent $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$. Take $\mathbb{Z}^2 = \langle a, b \rangle \curvearrowright \mathbb{R}/\mathbb{Z}$ by $a \cdot x = x + \alpha$ and $b \cdot x = x + \beta$.





Proper coloring is an LCL

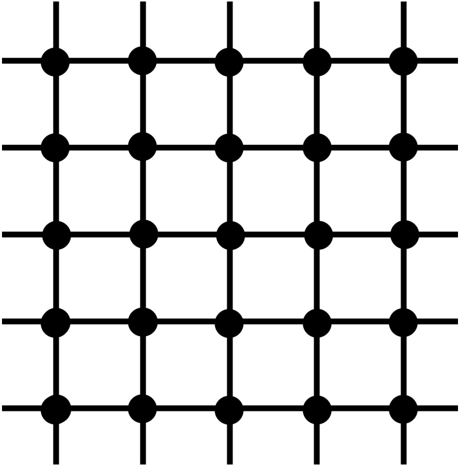


Figure: Proper coloring on $\mathbb{Z}^2$

Proper coloring is an LCL

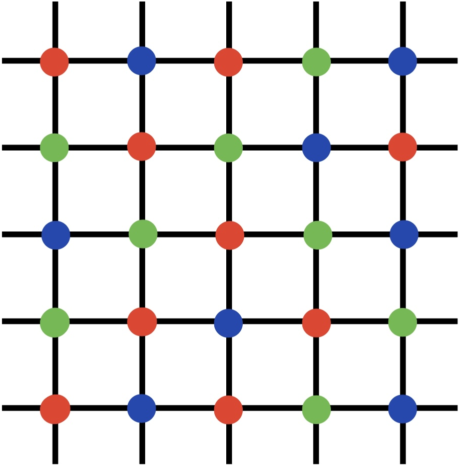


Figure: Proper coloring on $\mathbb{Z}^2$

Proper coloring is an LCL

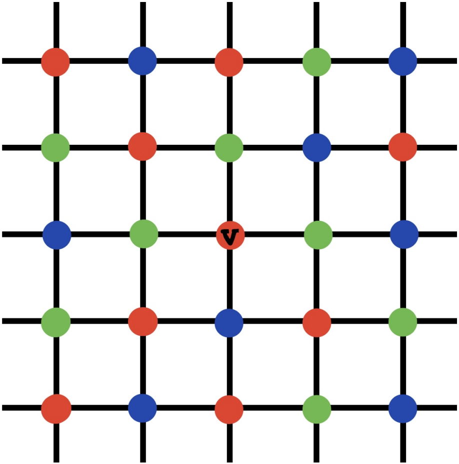


Figure: Proper coloring on $\mathbb{Z}^2$

Proper coloring is an LCL

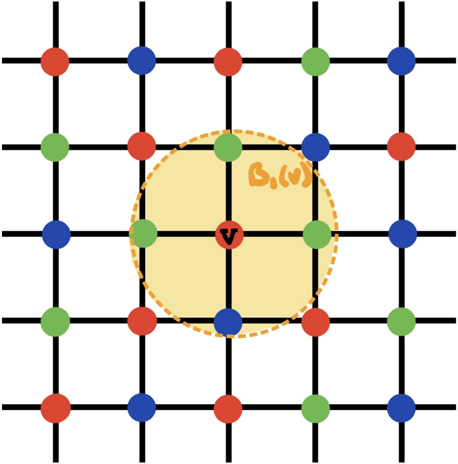


Figure: Proper coloring on $\mathbb{Z}^2$

Proper coloring is an LCL

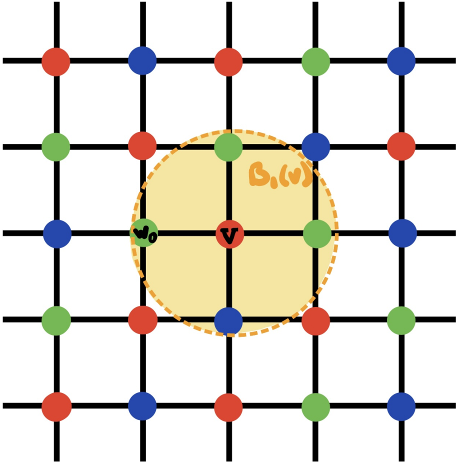


Figure: Proper coloring on $\mathbb{Z}^2$

Proper coloring is an LCL

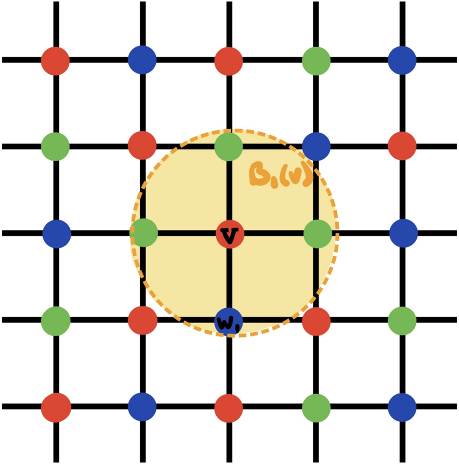


Figure: Proper coloring on $\mathbb{Z}^2$

Proper coloring is an LCL

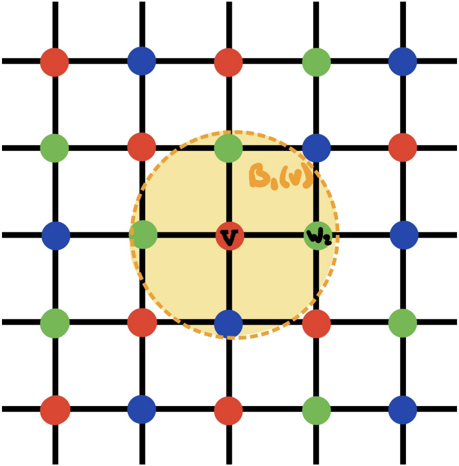


Figure: Proper coloring on $\mathbb{Z}^2$

Proper coloring is an LCL

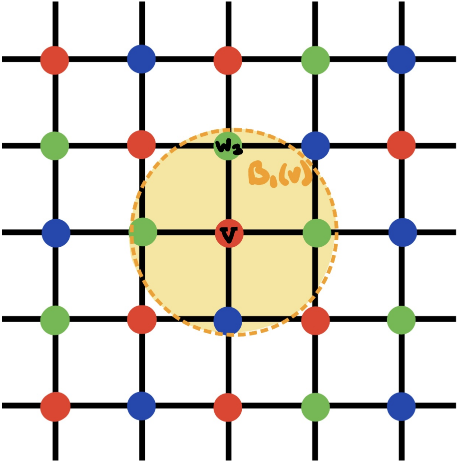


Figure: Proper coloring on $\mathbb{Z}^2$

Proper coloring is an LCL

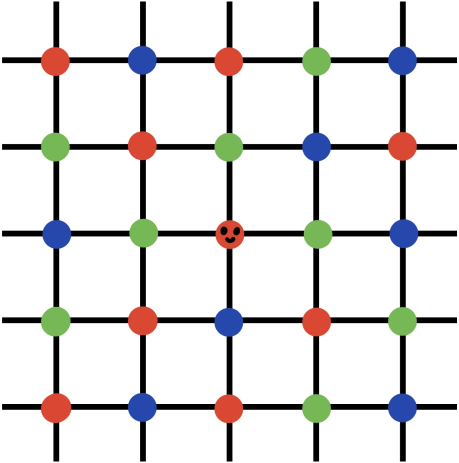


Figure: Proper coloring on $\mathbb{Z}^2$

Proper coloring is an LCL

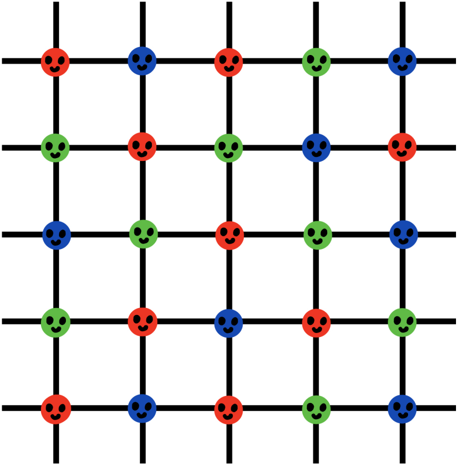


Figure: Proper coloring on $\mathbb{Z}^2$

Locally Checkable Labeling Problems

Idea: Label the vertices of G according to some *rule* which can be verified *locally*.

Examples of LCLs:

- proper vertex coloring
- proper edge coloring
- matchings
- sinkless orientation
- Wang tiling

Nonexamples of LCLs:

- Hamiltonian cycle
- spanning trees

Table of Contents

Different Notions of Definability

Definition (Descriptive complexity classes)

For an LCL Π, we say:

- $\Pi \in \mathrm{BOREL}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mathcal{B})$ on a standard Borel space admits a **Borel** solution.
- $\Pi \in \mathrm{MEAS}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mu)$ on a standard probability space admits a μ-**measurable** solution.
- $\Pi \in \mathrm{BAIRE}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \tau)$ on a Polish space admits a **Baire measurable** solution.

Different Notions of Definability

Definition (Descriptive complexity classes)

For an LCL Π, we say:

- $\Pi \in \text{BOREL}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mathcal{B})$ on a standard Borel space admits a **Borel** solution.
- $\Pi \in \text{MEAS}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mu)$ on a standard probability space admits a μ-**measurable** solution.
- $\Pi \in \text{BAIRE}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \tau)$ on a Polish space admits a **Baire measurable** solution.

Different Notions of Definability

Definition (Descriptive complexity classes)

For an LCL Π, we say:

- $\Pi \in \mathsf{BOREL}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mathcal{B})$ on a standard Borel space admits a **Borel** solution.
- $\Pi \in \mathsf{MEAS}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mu)$ on a standard probability space admits a μ-**measurable** solution.
- $\Pi \in \mathsf{BAIRE}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \tau)$ on a Polish space admits a **Baire measurable** solution.

Different Notions of Definability

Definition (Descriptive complexity classes)

For an LCL Π, we say:

- $\Pi \in \mathrm{BOREL}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mathcal{B})$ on a standard Borel space admits a **Borel** solution.
- $\Pi \in \mathrm{MEAS}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mu)$ on a standard probability space admits a μ-**measurable** solution.
- $\Pi \in \mathrm{BAIRE}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \tau)$ on a Polish space admits a **Baire measurable** solution.

Example: Proper $2n$-coloring is in $\mathrm{MEAS}(\mathbb{F}_n)$ and $\mathrm{BAIRE}(\mathbb{F}_n)$ by Conley–Marks–Tucker-Drob (2016) but not in $\mathrm{BOREL}(\mathbb{F}_n)$ by Marks (2013).

Complexity Classes

BOREL, MEAS, BAIRE, FIID, FFIID, . . .

Complexity Classes

BOREL, MEAS, BAIRE, FIID, FFIID, . . .

Question: Are all of these classes distinct? What inclusions can we establish?

Complexity Classes

BOREL, MEAS, BAIRE, FIID, FFIID, . . .

Question: Are all of these classes distinct? What inclusions can we establish?

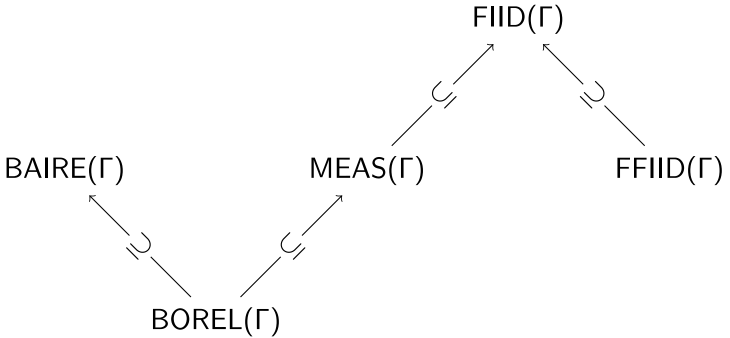


Figure: The trivial inclusions for any Γ.

Previous Results

Grebík–Rozhoň (2021) have shown:

$$\mathsf{BOREL}(\mathbb{Z}) = \mathsf{BAIRE}(\mathbb{Z}) = \mathsf{MEAS}(\mathbb{Z}) = \mathsf{FIID}(\mathbb{Z}) = \mathsf{FFIID}(\mathbb{Z})$$

Previous Results

Grebík–Rozhoň (2021) have shown:

$$\mathsf{BOREL}(\mathbb{Z}) = \mathsf{BAIRE}(\mathbb{Z}) = \mathsf{MEAS}(\mathbb{Z}) = \mathsf{FIID}(\mathbb{Z}) = \mathsf{FFIID}(\mathbb{Z})$$

Conley–Marks–Tucker-Drob (2016), Marks (2013), Bernshteyn and Brandt–Chang–Grebík–Grunau–Rozhoň–Vidnyánszky (2021), Conley–Miller (2011), and Conley–Kechris (2013) have shown:

$$\mathsf{BOREL}(\mathbb{F}_2) \subsetneq \mathsf{MEAS}(\mathbb{F}_2) \subsetneq \mathsf{BAIRE}(\mathbb{F}_2)$$

Previous Results

Grebík–Rozhoň (2021) have shown:

$$\mathrm{BOREL}(\mathbb{Z}) = \mathrm{BAIRE}(\mathbb{Z}) = \mathrm{MEAS}(\mathbb{Z}) = \mathrm{FIID}(\mathbb{Z}) = \mathrm{FFIID}(\mathbb{Z})$$

Conley–Marks–Tucker-Drob (2016), Marks (2013), Bernshteyn and Brandt–Chang–Grebík–Grunau–Rozhoň–Vidnyánszky (2021), Conley–Miller (2011), and Conley–Kechris (2013) have shown:

$$\mathrm{BOREL}(\mathbb{F}_2) \subsetneq \mathrm{MEAS}(\mathbb{F}_2) \subsetneq \mathrm{BAIRE}(\mathbb{F}_2)$$

The following was left open:

- (Grebík–Rozhoň) Is $\mathrm{BOREL}(\mathbb{Z}^n) \subseteq \mathrm{MEAS}(\mathbb{Z}^n)$ strict for $n > 1$?
- Does $\mathrm{MEAS}(\Gamma) \subseteq \mathrm{BAIRE}(\Gamma)$ hold for all Γ?
- Is $\mathrm{FFIID}(\Gamma) = \mathrm{FIID}(\Gamma)$ for every Γ?

Table of Contents

Problems on $\mathbb{Z}^d$

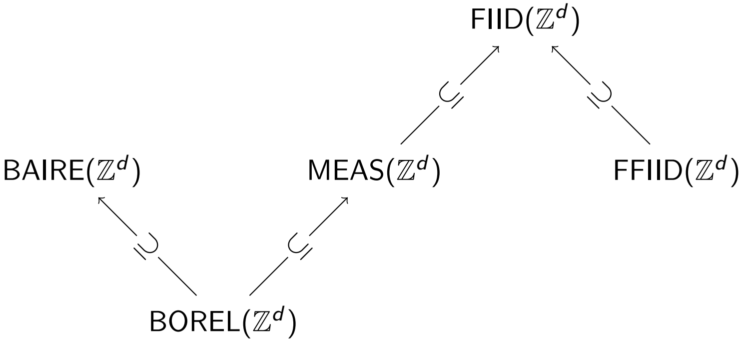


Figure: Complexity classes of LCLs on $\mathbb{Z}^d$, $d \geq 2$.

Problems on $\mathbb{Z}^d$

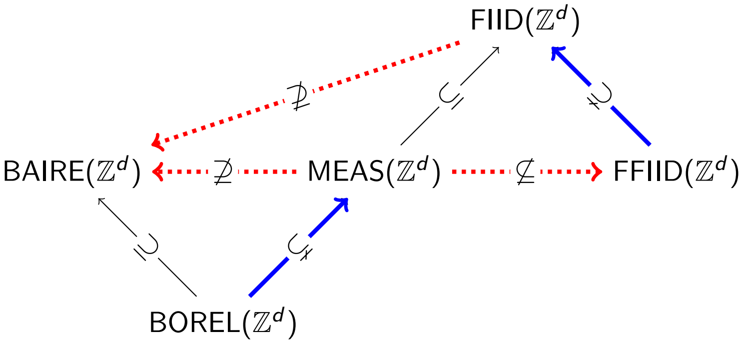


Figure: Complexity classes of LCLs on $\mathbb{Z}^d$, $d \geq 2$.

Blue arrows are strict inclusions. $\subsetneq$
Red dotted arrows are noninclusion. $\not\subseteq$

Our Results

Theorem (B.–Bernshteyn–Lyons–Weilacher)

For $d \geq 2$, there is an LCL Π on $\mathbb{Z}^d$ so that:

- $\Pi \in \mathrm{MEAS}(\mathbb{Z}^d)$,
- $\Pi \notin \mathrm{BAIRE}(\mathbb{Z}^d)$,
- $\Pi \in \mathrm{FIID}(\mathbb{Z}^d)$,
- $\Pi \notin \mathrm{FFIID}(\mathbb{Z}^d)$.

Our Results

Theorem (B.–Bernshteyn–Lyons–Weilacher)

For $d \geq 2$, there is an LCL Π on $\mathbb{Z}^d$ so that:

- $\Pi \in \mathrm{MEAS}(\mathbb{Z}^d)$,
- $\Pi \notin \mathrm{BAIRE}(\mathbb{Z}^d)$,
- $\Pi \in \mathrm{FIID}(\mathbb{Z}^d)$,
- $\Pi \notin \mathrm{FFIID}(\mathbb{Z}^d)$.

This is the first example of a group Γ with $\mathrm{MEAS}(\Gamma) \not\subseteq \mathrm{BAIRE}(\Gamma)$ and the first group with $\mathrm{FFIID}(\Gamma) \neq \mathrm{FIID}(\Gamma)$.

Table of Contents

Toast

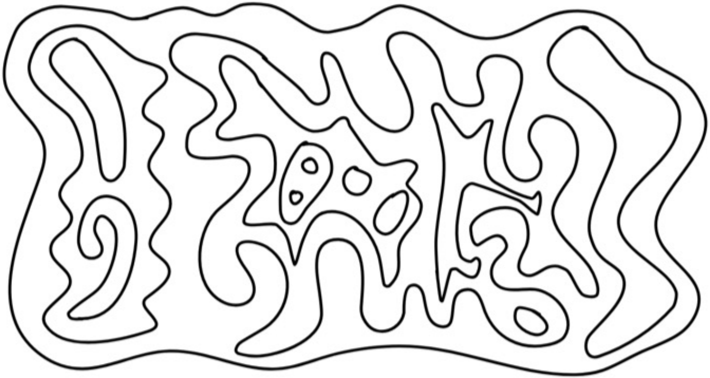


Figure 1. A few pieces of a toast.

Toast

Definition

Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action inducing a graph G. We say that a collection of finite sets $\mathcal{T} \subseteq [X]^{<\omega}$ with $\bigcup \mathcal{T} = X$ is a Borel q-**toast** if the following two conditions hold for all $K, L \in \mathcal{T}$,

- either $K \cap L = \varnothing$, $K \subseteq L$, or $L \subseteq K$,
- we have $d(\partial K, \partial L) \geq q$ in the graph metric.

Note: Borel graphs are hyperfinite iff they admit a 0-toast.

Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):

Borel graphs induced by free actions of $\mathbb{Z}^d$ on a standard Borel space admit a Borel q-toast for any $q \in \mathbb{N}$.

Why toast?

Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):

Borel graphs induced by free actions of $\mathbb{Z}^d$ admit a Borel proper 3-coloring.

Why toast?

Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):

Borel graphs induced by free actions of $\mathbb{Z}^d$ admit a Borel proper 3-coloring.

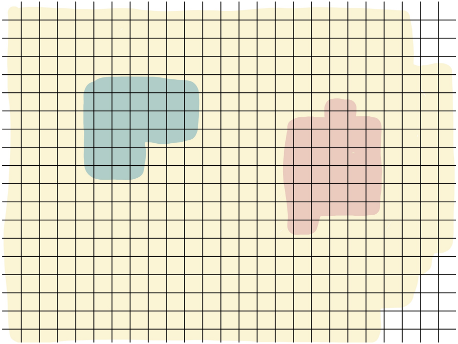


Figure: Proper coloring on $\mathbb{Z}^2$

Why toast?

Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):

Borel graphs induced by free actions of $\mathbb{Z}^d$ admit a Borel proper 3-coloring.

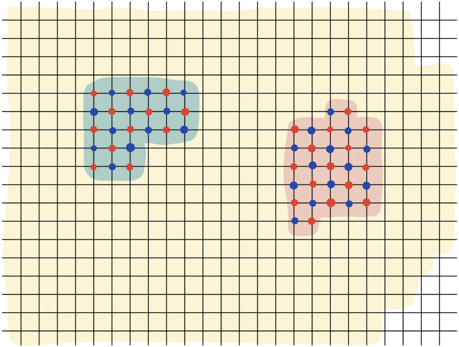


Figure: Proper coloring on $\mathbb{Z}^2$

Why toast?

Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):

Borel graphs induced by free actions of $\mathbb{Z}^d$ admit a Borel proper 3-coloring.

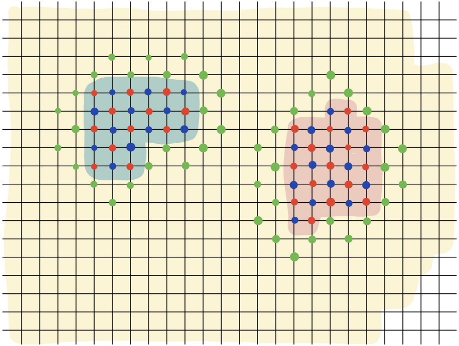


Figure: Proper coloring on $\mathbb{Z}^2$

Why toast?

Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):

Borel graphs induced by free actions of $\mathbb{Z}^d$ admit a Borel proper 3-coloring.

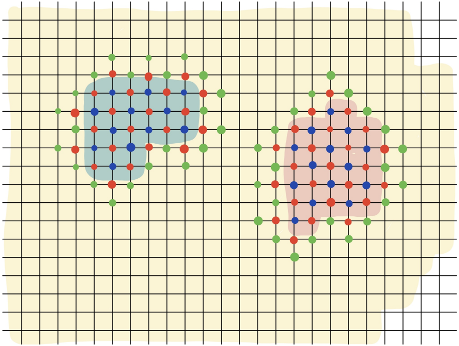


Figure: Proper coloring on $\mathbb{Z}^2$

Why toast?

Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):

Borel graphs induced by free actions of $\mathbb{Z}^d$ admit a Borel proper 3-coloring.

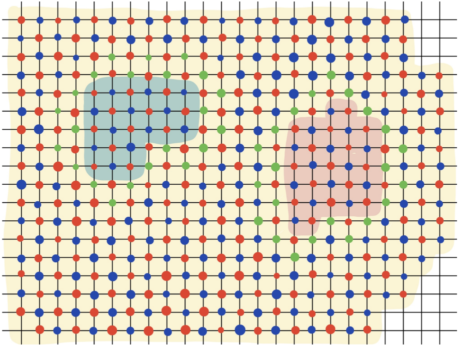


Figure: Proper coloring on $\mathbb{Z}^2$

Rectangular Toast

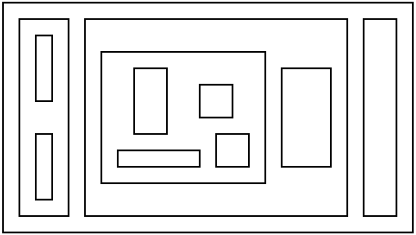


Figure 3. A rectangular toast for $\mathbb{Z}^2$.

Definition

A **rectangular** q-toast is a q-toast whose pieces are all rectangles.

Rectangular Toast

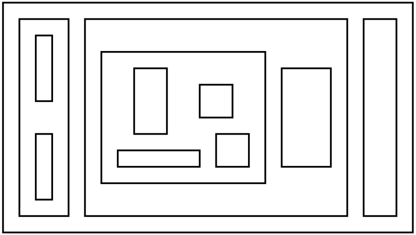


Figure 3. A rectangular toast for $\mathbb{Z}^2$.

Definition

A **rectangular** q-toast is a q-toast whose pieces are all rectangles.

Theorem (folklore):

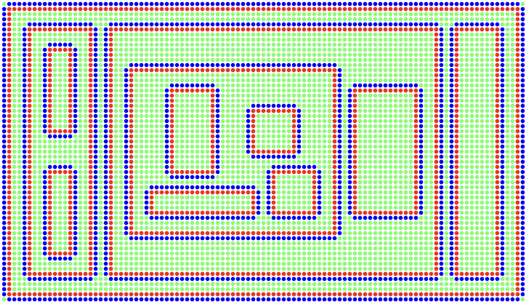
Free Borel actions of $\mathbb{Z}^d$ on a standard probability space admit rectangular q-toast on a conull set (but not on a comeager set).

Naive Attempt

What if we try to encode rectangular toast as an LCL?

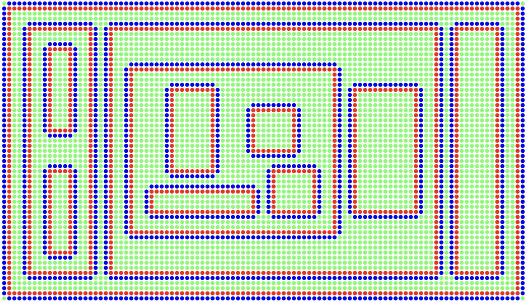
Naive Attempt

What if we try to encode rectangular toast as an LCL?



Naive Attempt

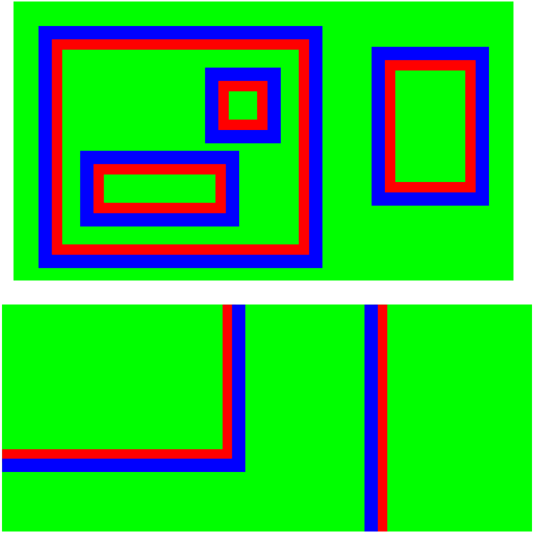
What if we try to encode rectangular toast as an LCL?



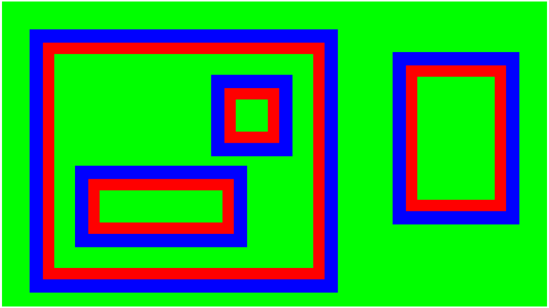
Consider the LCL whose solutions 3-color $\mathbb{Z}^d$ so it *locally* looks like the picture above.

Issues with this

These diagrams locally look the same.

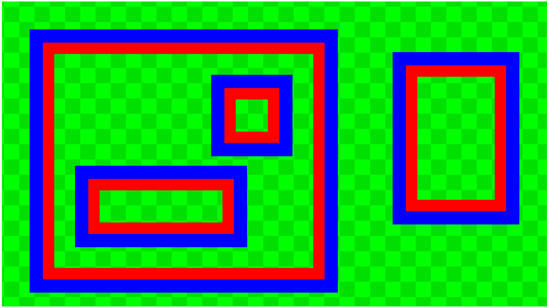


The Fix



The Fix

Require the green regions to be 2-colored. This is our new LCL CRT.



We then have CRT $\in$ MEAS($\mathbb{Z}^d$) by the existence of a rectangular toast.

CRT has no Baire Measurable Solution

Theorem (B.–Bernshteyn–Weilacher–Lyons):

CRT does not always admit a Baire measurable solution.

Proof.

- Let $Z^d \curvearrowright X$ be an appropriate action. Assume for contradiction $f : X \to \{\text{RED}, \text{BLUE}, \text{GREEN0}, \text{GREEN1}\}$ be a Baire measurable solution to CRT. Let $\mathcal{T}$ be the corresponding (possibly partial) rectangular toast encoded by f.

CRT has no Baire Measurable Solution

Theorem (B.–Bernshteyn–Weilacher–Lyons):

CRT does not always admit a Baire measurable solution.

Proof.

- Let $Z^d \curvearrowright X$ be an appropriate action. Assume for contradiction $f : X \to \{\text{RED}, \text{BLUE}, \text{GREEN0}, \text{GREEN1}\}$ be a Baire measurable solution to CRT. Let $\mathcal{T}$ be the corresponding (possibly partial) rectangular toast encoded by f.
- Consider the generic orbit $\mathcal{O}$, which we show does not admit complete rectangular toast. Therefore we can show $X \setminus \bigcup \mathcal{T}$ is connected.

CRT has no Baire Measurable Solution

Theorem (B.–Bernshteyn–Weilacher–Lyons):

CRT does not always admit a Baire measurable solution.

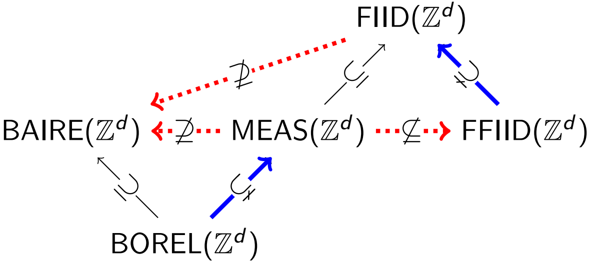
Proof.

- Let $Z^d \curvearrowright X$ be an appropriate action. Assume for contradiction $f : X \to \{\text{RED}, \text{BLUE}, \text{GREEN0}, \text{GREEN1}\}$ be a Baire measurable solution to CRT. Let $\mathcal{T}$ be the corresponding (possibly partial) rectangular toast encoded by f.
- Consider the generic orbit $\mathcal{O}$, which we show does not admit complete rectangular toast. Therefore we can show $X \setminus \bigcup \mathcal{T}$ is connected.
- Then, f can be extended uniquely to a Baire measurable 2-coloring. Contradiction.

□

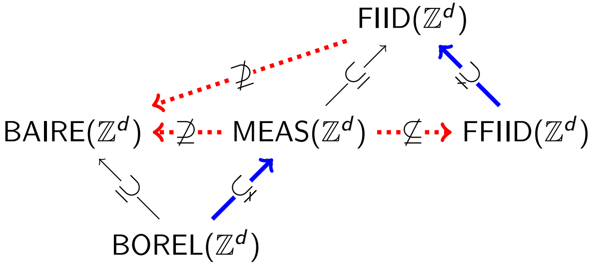
Table of Contents

Open Questions



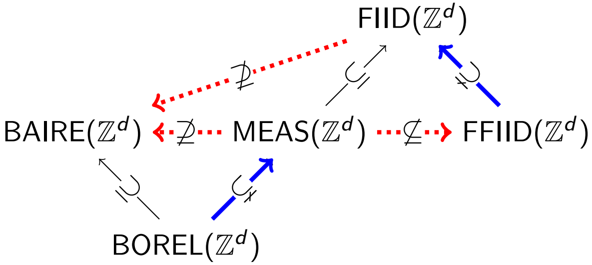
Some arrows are still missing.

Open Questions



Some arrows are still missing. Lets make it a complete graph!

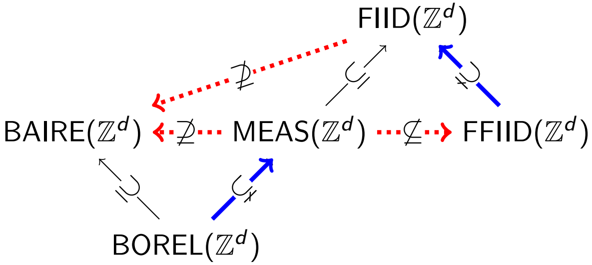
Open Questions



Some arrows are still missing. Lets make it a complete graph!

Question: Does BAIRE($\mathbb{Z}^d$) $=$ BOREL($\mathbb{Z}^d$)?

Open Questions

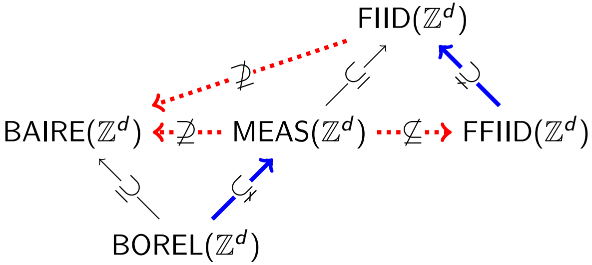


Some arrows are still missing. Lets make it a complete graph!

Question: Does $\mathsf{BAIRE}(\mathbb{Z}^d) = \mathsf{BOREL}(\mathbb{Z}^d)$?

Question: Does $\mathsf{MEAS}(\mathbb{Z}^d) = \mathsf{FIID}(\mathbb{Z}^d)$?

Open Questions



Some arrows are still missing. Lets make it a complete graph!

Question: Does BAIRE($\mathbb{Z}^d$) $=$ BOREL($\mathbb{Z}^d$)?

Question: Does MEAS($\mathbb{Z}^d$) $=$ FIID($\mathbb{Z}^d$)?

Question: What is the relationship between BOREL($\mathbb{Z}^d$) and FFIID($\mathbb{Z}^d$)?

Thanks!