Amenable actions on trees

Katalin Berlow

December 12, 2024

Contents

0 Introduction

In this note we wish to discuss a result about the actions of amenable groups on trees. Recall that a group acts freely on a tree if and only if the group itself is free. So, if a group is not free, any action of that group on a tree will have a nontrivial element which fixes a point. Amenability can be viewed as a strengthening of not containing a nonabelian free group. So, in fact, we have a stronger result for the actions of amenable groups: we have that there are certain sets which are fixed *globally* by the entire action. The goal of this note is to write up the proof of the following theorem:

Theorem 0.0. *Let Γ be an amenable group acting on a tree T by automorphisms. Then the action fixes one of the following:*

- *a vertex*
- *an edge (possibly flipping)*
- *an end*
- *a pair of ends (possibly flipping).*

Towards this we will review some theory on Boolean algebras, Stone duality, ends of groups, and the Helly property for trees. Not all of the reviewed material will be directly necessary for the proof of Theorem 0.0, but will motivate the techniques used.

1 Background on Boolean algebras

Recall the following definitions.

Definition 1.0 (Lattice, order-theoretic)**.** A *lattice* is a poset equipped with two binary relations: join (denoted $\vee$) and meet (denoted $\wedge$) which satisfy the following monotonicity property:

$$\text{If } a_1 \leq a_2 \text{ and } b_1 \leq b_2 \text{ then } a_1 \vee b_1 \leq a_2 \vee b_2 \text{ and } a_1 \wedge b_1 \leq a_2 \wedge b_2.$$

The fallowing is an alternative, more algebraic definition of a lattice:

Definition 1.1 (lattice, algebraic)**.** A Lattice is a set L equipped with two binary, commutative and associative operations $\vee$ and $\wedge$ such that the following absorption laws hold:

$$\text{For all } a, b \in L \text{ we have } a \vee (a \wedge b) = a \text{ and } a \wedge (a \vee b) = a.$$

These two definitions are equivalent. The following are examples of lattices:

Example 0. For any set A the power set of A, ordered by set inclusion, is a poset. Along with union as join and intersection as meet, this becomes a lattice.

Example 1. The set of divisors of n ordered by divisibility is a poset. Along with least common multiple as join and greatest common divisor as meet, it becomes a lattice.

Example 2. For any set A, the set of partitions of A ordered by refinement is a poset. For partitions a and b of A, we let $a \vee b$ be the partition where for $x, y \in A$ we have $x \sim_{a \vee b} y$ iff $x \sim_a y$ or $a \sim_n y$. We analogously let $a \wedge b$ be the partition where for $x, y \in A$ we have $x \sim_{a \wedge b} y$ iff $x \sim_a y$ and $a \sim_n y$. This forms a lattice.

Definition 1.2 (Distributed Lattice)**.** A lattice $(L, \vee, \wedge)$ is *distributive* if the following holds:

$$\text{For all } a, b, c \in L \text{ we have } a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Note that Example 0 and Example 1 are both distributive lattices. The lattice in Example 2 is not in general.

Definition 1.3 (Complemented Lattice)**.** A lattice L is *complemented* iff for every $a \in L$ there is a $b \in L$, called the complement, where $a \vee b = 1$ and $a \wedge b = 0$.

Definition 1.4 (Boolean Algebra)**.** A *Boolean algebra* is a complemented distributed lattice

Note that Example 0 is a complemented lattice, with set complement acting as the complement. For Example 1, the set of divisors of n form a Boolean algebra iff n is square-free.

It can be shown that every finite Boolean algebra is isomorphic to the Boolean algebra of all subsets of a finite set. As we will see in Theorem 2.5, every Boolean algebra A is isomorphic to the Boolean algebra of all clopen sets in some (compact totally disconnected Hausdorff) topological space.

Definition 1.5. A subset $U \subseteq B$ of a Boolean algebra B is an ultrafilter iff:

0. If $b \in u$ and $a \leq b$ then $a \in U$,
1. $a \wedge b \in B$ if $a, b \in U$.
2. For all $a \in B$ either $a \in U$ or $\neg a \in U$ where $\neg a$ is the complement of a.

Note that condition 2 is equivalent to requiring that U be maximal. Without condition 2, we call U a *filter*.

2 Stone Duality

Given a Boolean algebra B, we can associate to it a topological space $S(B)$, called its *stone space*.

Definition 2.0 (Stone Space)**.** Let B be a Boolean algebra. The *stone space* of B is the topological space consisting of the set $S(B)$ of ultrafilters on B equipped with the topology generated by basic open sets of the form

$$N_b := \{U \in S(B) : b \in U\}$$

for each $b \in B$.

Stone spaces are actually quite nice topological spaces.

Theorem 2.1. *Stone spaces are compact, zero-dimensional, Hausdorff spaces.*

We will prove this through three lemmas.

Lemma 2.2. *Stone spaces are zero-dimensional.*

Proof. Let B be a Boolean algebra. Note that for any element $a \in B$, we have that

$$S(B) \setminus N_a := N_{\neg a}$$

is open, where $\neg a$ is the complement of a. Thus, N_a is clopen for any $a \subseteq B$. ■

Lemma 2.3. *Stone spaces are Hausdorff.*

Proof. Let B be a Boolean algebra and U, U' distinct ultrafilters on B. Since they are not equal, there must be an element of B on which they disagree. Without loss of generality, let $a \in B$ be so $a \in U$ but $a \notin U'$. Then we have that N_a is an open set such that $U \in N_a$ but $U' \notin N_a$. ■

Lemma 2.4. *Stone spaces are compact.*

Proof of Lemma 2.4. Let B be a Boolean algebra.

Let $\{N_{a_i}\}_{i \in \mathbb{N}}$ be a family of basis elements for $S(B)$. Suppose that it has no finite subcover. Then for any $a_0, \ldots, a_n$ there is an ultrafilter containing none of these. This implies that the collection $\neg a_i$ generates a proper filter and can be extended to an ultrafilter U by Zorn's lemma. This U witnesses that the original family was not a cover of the Stone space. ■

Proof of Theorem 2.1. This follows from Lemma 2.2, Lemma 2.3, and Lemma 2.4. ■

The converse of Theorem 2.1 also holds.

Theorem 2.5. *Let X be a compact, zero-dimensional, Hausdorff space. Then there is a Boolean algebra B so $X \cong S(B)$.*

Proof. Let B be the Boolean algebra of all clopen subsets of X. Define a homeomorphism $\varphi : X \to S(B)$ by sending $x \in X$ to the set $\mathcal{U}_x := \{A \in B : x \in A\}$.

To see this is bijective, we will show that for any ultrafilter $\mathcal{U} \in S(B)$, there is an $x \in X$ such that $\bigcap \mathcal{U} = \{x\}$. Since $\mathcal{U}$ is a proper filter, it has the finite intersection property. So, since the element of $\mathcal{U}$ are clopen, and X is compact, we have that $\bigcap \mathcal{U} \neq \varnothing$ by the finite intersection property. To see that there is at most one point in $\bigcap \mathcal{U}$, for contradiction, let $x, y \in \bigcap \mathcal{U}$. Since X is Hausdorff and zero-dimensional, we can find disjoint, clopen B_x, B_y with $x \in B_x$, $y \in B_y$. But, since $(X \setminus B_x) \cup (X \setminus B_y) = X$ either $X \setminus B_x$ or $X \setminus B_y$ is in the ultrafilter. This contradicts $x, y \in \bigcap \mathcal{U}$.

This map is continuous since if $N_a \subseteq S(B)$ is a basic clopen set, then $\varphi^{-1}(N_a) = \bigcap \{\mathcal{U} \in X : U \in N_a\}$. Note that $a \in \mathcal{U}_x$ iff $x \in a$. So, $\varphi^{-1}(N_a) = \bigcap \{\mathcal{U}_x \in X : x \in a\} = a$, which is clopen. So, since this continuous map is from a compact space to a Hausdorff space, it is automatically open and thus a homeomorphism. ■

3 Ends of Trees

In this section we will review the necessary background on the space of ends of a tree.

Definition 3.0. Let $T = (V, E)$ be a tree. A *geodesic path* in T is a sequence of vertices $(v(i))_{i \in I}$ where for any $n, m \in I$, we have $d(v(n), v(m)) = |n - m|$. Here, we let I be $\mathbb{Z}, \mathbb{N}$, or $[n]$ for $n \in \mathbb{N}$.

Note that if $I = \mathbb{Z}$, the geodesic is a bi-infinite path, if $I = \mathbb{N}$ it is a ray, and if $I = [n]$, it is a finite path.

Definition 3.1. Let $T = (V, E)$ be a tree and $\mathcal{R}$ is the set of geodesic rays. Define an equivalence relation $\sim$ on $\mathcal{R}$ by letting $r \sim r'$ iff r and r' are eventually the same. That is, if there are $n, n' \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ $r(n+i) = r(n'+i)$. Then, the *space of ends* of T, denoted ∂T, is the set $\mathcal{R}/\sim$. We let $\overline{T} := V \cup \partial_T V$

An element of this quotient should be viewed as an infinite branch through the tree, forgetting about the starting point.

Let $x \in V$ and $\xi \in \partial_T V$. We say that the geodesic $\{v(n)\}_{n\in\mathbb{N}}$ *is from* x to ξ iff $v(0) = x$ and $\{v(n)\}_{n\in\mathbb{N}}$ is in the equivalence class ξ. If $\xi, \eta \in \partial_T V$, then the geodesic $\{v(n)\}_{n\in\mathbb{Z}}$ connects ξ to η iff $\{v(i)\}_{i\in\mathbb{N}}$ is a representative for ξ and $\{v(-i)\}_{i\in\mathbb{N}}$ is a representative for η. Observe that such a geodesic is unique (up to indexing in the case of $\xi, \eta \in \partial_T V$).

We let $[x, y]$ denote the set of vertices lying on the unique geodesic from x to y. Observe that we have a sort of hyperbolicity condition on $\overline{T}$: if $x, y, z \in \overline{T}$ then the geodesic $[x, y] \cap [y, z] \cap [z, x]$ contains exactly one vertex, called the *median* of x, y and z.

We are now ready to define the topology on $\overline{T}$. We wish to equip $\overline{T}$ with the topology generated by halfspaces. Note that given an edge (x, y) of T, by deleting this edge, we split T into two connected components: one containing x and one containing y. We call these two sets *halfspaces.* This leads us to the following definition.

Definition 3.2. The *end completion* of T, is the space $\overline{T}$ equipped with the following topology. Let $x \in \overline{T}$ and $F \subset E$ be a finite subset of edges. Then, let

$$U(x, F) = \{y \in \overline{T} : [x, y] \cap F = \varnothing\}.$$

The topology on $\overline{T}$ is that generated by the open sets $U(x, F)$ for $x \in \overline{T}$ and finite $F \subset T$.

The set $U(x, F)$ is the connected component of x after deleting the finitely many edges in F. Alternatively, this is intersection of the halfspaces associated to x and each edge in F.

Note that the sets $U(x, F)$ as $x \in \overline{T}$ and finite $F \subseteq E$ vary forms a basis for our topology: If $x \in U(x_1, F_1) \cap U(x_2, F_2)$, then we also have $x \in U(x, F_1 \cup F_2) \subseteq U(x_1, F_1) \cap U(x_2, F_2)$.

Note that the topology on $\overline{T}$ is Hausdorff since if $x \neq y$ and $z \in [x, y]$ then $U(x, \{z\}) \cap U(y, \{z\}) = \varnothing$. If x and y are adjacent, we alternatively have $U(x, \{y\}) \cap U(y, \{x\}) = \varnothing$.

Lemma 3.3. *A vertex is isolated iff it has finite degree.*

Proof. If x has finite degree, then $U(x, N(x)) = \{x\}$. If x is isolated, there is some basic open set

$$U(x_0, F_0) = \{x\}.$$

We must have that $N(x) \subseteq F_0$, otherwise if $y \in N(x) \setminus F_0$, then $y \in U(x_0, F_0)$. ■

We will now take a detour to orientations of trees and how they provide a bridge between ultrafilters and elements of the end completion.

Definition 3.4. An orientation $\mathcal{O}$ on a tree T is a choice of direction for each of the edges. That is, for each $e = (x, y) \in E$ it there is a designated $o(e) \neq t(e)$ with $o(e), t(e) \in \{x, y\}$

Definition 3.5. Let T be a tree and $\mathcal{O}$ an orientation of T. We call $\mathcal{O}$ *coherent* if there is no vertex of T with two outgoing edges.

Lemma 3.6. *Let $\mathcal{O}$ be a coherent orientation on T. Then all edges point to a vertex or an end.*

Proof. When starting at a vertex and following the outgoing edges, either one will go towards an end, or will reach a vertex with no outgoing edges. Note that if two such vertices exist, the path between them cannot be coherent. ■

Let $\mathcal{B}(T)$ denote the boolean algebra generated by the halfspaces in T. This is the algebra of all subsets of V with finite edge boundary.

Lemma 3.7. *Let T be a tree and $\mathcal{U}$ an ultrafilter on $\mathcal{B}(T)$. Then, there is coherent orientation $\mathcal{O}_\mathcal{U}$ on T such that:*

- *$\mathcal{U}$ is a principal ultrafilter determined by a vertex v iff all edges of $\mathcal{O}_\mathcal{U}$ point to v,*
- *$\mathcal{U}$ is a nonprincipal ultrafilter iff all edges of $\mathcal{O}_\mathcal{U}$ point to an end ξ where any element of $\mathcal{U}$ contains ξ.*

Proof. Let $\mathcal{U}$ be an ultrafilter on the set of subtrees of T. For each edge (x, y), let S_x and S_y be the two halfspaces created by deleting (x, y). Since $\mathcal{U}$ is an ultrafilter, either $S_x \in \mathcal{U}$ or $S_y \in \mathcal{U}$. Direct the edge (x, y) towards the halfspace contained in the ultrafilter. This will be an orientation since we can never have both halfspaces in the ultrafilter. This orientation will be coherent since two edges pointing away from each other gives us two disjoint elements of the ultrafilter, which is not possible.

By Lemma 3.6, the orientation points towards a vertex or an end. If the orientation will points towards a single vertex, then it is clear that every set containing that vertex is in the ultrafilter. If the orientation points towards an end ξ, then any element of $\mathcal{U}$ must have an ■

Theorem 3.8. *Let T be a tree. Then the associated stone space $S(\mathcal{B}(T))$ is homeomorphic to $\overline{T}$, the end completion of T.*

Proof. As shown in Lemma 3.7, each ultrafilter in $S(\mathcal{B}(T))$ gives us an orientation which point to either a vertex or an end. So, this gives rise to an association between $S(\mathcal{B}(T))$ and the end completion of T. Call this map φ.

We will show that φ is continuous. Let $H(x, y)$ be the halfspace in $\overline{T}$ obtained by deleting the edge (y, x) and containing x. Note that if U is an ultrafilter, then we have that $\varphi(\mathcal{U}) \in H(y, x)$ if and only if the corresponding orientation $\mathcal{O}_\mathcal{U}$ (defined by Lemma 3.7) orients the edge (y, x) towards x. This is because, if that edge is oriented towards x, because the orientation must be coherent, the vertex or end it points to must be in $H(y, x)$. The condition that $\mathcal{O}_\mathcal{U}$ orients the edge (y, x) towards x is equivalent to requiring that $H(y, x) \in \mathcal{U}$. So, we have that $\varphi^{-1}(H(y, x)) = N_{H(y,x)}$.

To see that φ is open, note that $x \in \varphi(N_a)$ if and only if the principle ultrafilter defined by x contains the set a. All this is saying is that $x \in a$. As a is a set with finite edge boundary, it is open. So, $x \in \varphi(N_a) = a$ and φ is open. ■

This leads to the following immediate corollary.

Corollary 3.9. *Let Γ be a countable group acting by automorphisms on a tree T. This action can be extended to a continuous action $\Gamma \curvearrowright \overline{T}$ on the end completion of T.*

4 Proof of Main Theorem

We will now prove the theorem:

Theorem 0.0. *Let Γ be an amenable group acting on a tree T by automorphisms. Then the action fixes one of the following:*

- *a vertex*
- *an edge (possibly flipping)*
- *an end*
- *a pair of ends (possibly flipping).*

Proof. Let $\Gamma \curvearrowright T$ by automorphisms. By Corollary 3.9, this action can be extended to a continuous action on the end completion $\overline{T}$. Since Γ is amenable and $\overline{T}$ is compact, this action admits an invariant countably additive probability measure, μ.

Let $H(x,y)$ denote the halfspace of T containing y and not x. Consider the set A of edges (x,y) such that the halfspaces $H(x,y)$ and $H(y,x)$ are each measure 1/2. Note that A is invariant under the action of G. We also have that A is connected since if $x_0, \dots x_n$ is a geodesic path in T with (x_0,x_1) and (x_{n-1},x_n) in A, then we have that $\mu(H(x_0,x_1)) = \mu(H(x_{n-1},x_n)) = 1/2$ and $H(x_{n-1},x_n) \subseteq H(x_0,x_1)$, we then have $\mu(H(x_i,x_{i+1})) = 1/2$ as well (since $H(x_0,x_1) \subseteq H(x_i,x_{i+1}) \subseteq H(x_n,x_{n+1})$)for $i < n$. So, the path is contained in A.

We will show that every vertex in the sub graph induced by A has degree at most 2. Assume for contradiction that $(x_0,y),(x_1,y),(x_2,y)$ are edges in A. Then, we must have that $H(x_0,y)$ and $H(y,x_0)$ have measure 1/2. But, then the sets $H(y,x_0)$,$H(y,x_1)$,$H(y,x_2)$ are disjoint and each have measure $\frac{1}{2}$. This is a contradiction.

Since A has degree at most 2, there are four cases:

Case 1: A is finite and nonempty. In this case, we must have that the action of Γ either fixes A or flips A. Either way, there is a fixed vertex or edge of A and we are done.

Case 2: A is an infinite ray. In this case, the base point of the ray is fixed.

Case 3: A is a bi-infinite line. Then the pair of ends represented by the rays are fixed set-wise under the action.

Case 4: A is empty. This gives us an invariant coherent orientation $\mathcal{O}_\mu$ of the tree, by directing each edge towards the halfspace of larger measure. As shown in Lemma 3.6, this orientation points towards a single vertex or end. Since this orientation is invariant, that vertex or end must be fixed by the action, as desired. ■