LINEAR ULTRAFILTERS

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ABSTRACT. Let X be a k-vector space, and U a maximal proper filter of subspaces of X. Then the ring of endomorphisms of X that are ''continuous'' with respect to U modulo the ideal of those that are ''trivial'' with respect to U forms a division ring $E(U)$. (These division rings can also be described as the endomorphism rings of the simple left $\mathrm{End}(X)$-modules.) We study this and the dual construction, based on maximal cofilters of subspaces of X; in particular, the relation between the constructed division rings and the original field or division ring k. We end by examining a more general construction in which X is a module over a general ring, given with both a filter and a cofilter of submodules.

1991 Mathematics Subject Classifications. Primary: 03C20, 15A04, 16K40; secondary: 03E50, 12L10, 16B70, 16D60, 16D90, 18E35.

This work was begun while Bergman was partly supported by NSF contract MCS 82-02632 and Hrushovski by an NSF graduate fellowship, and completed while Bergman was partly supported by NSF contract DMS 93-03379 and Hrushovski by DMS 94-00894. The authors are indebted to Mike May, whose notes from the Berkeley ''Topics in Algebra'' course in which some of these results arose helped us to reconstruct, 14 years later, what we had done.

1. Definitions and basic results.

Definition 1. *Let X be a vector space over a field k, or more generally, a right vector space over a division ring k.*

By a linear filter *on X we shall mean a nonempty set $\mathcal{F}$ of subspaces of X such that*

(i) *If $A \subseteq B$ are subspaces of X, then $A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$.*

(ii) *If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.*

A linear endomorphism f of X will be called continuous *with respect to the filter $\mathcal{F}$ if for all $A \in \mathcal{F}$, $f^{-1}(A) \in \mathcal{F}$, while f will be called* trivial *with respect to $\mathcal{F}$ if it satisfies the stronger condition $\ker(f) \in \mathcal{F}$. The ring (or if k is assumed commutative, the k-algebra) of endomorphisms of X continuous with respect to $\mathcal{F}$ will be denoted $R(\mathcal{F})$, the ideal of $R(\mathcal{F})$ consisting of endomorphisms trivial with respect to $\mathcal{F}$ will be denoted $I(\mathcal{F})$, and the factor-ring $E(\mathcal{F}) = R(\mathcal{F})/I(\mathcal{F})$ will be called the* eigenring *of $\mathcal{F}$.*

A linear filter will be called proper *if it does not contain all subspaces of X, equivalently, if it does not contain the subspace $\{0\}$. A maximal proper linear filter will be called a* linear ultrafilter *on X.*

The verification that for $\mathcal{F}$ a linear filter on X, $R(\mathcal{F})$ forms a unital subring of $\mathrm{End}(X)$, and $I(\mathcal{F})$ a 2-sided ideal of this subring, is straightforward. (We remark that throughout this note, linear maps among right vector spaces, and in §§7-9 among right modules, will be written to the left of their arguments, and composed accordingly.)

The definitions of linear filter and ultrafilter are, of course, inspired by the definitions of filter and ultrafilter of subsets of a set S [**9**, §7], [**6**, p.196], [**1**, Def. 1.3.1]. Now because every subset of a set S has a unique complement, if we dualize those set-theoretic concepts, we get nothing new: a ‘‘cofilter’’ of subsets of a set, defined in the manner dual to the definition of a filter of subsets, consists simply of the complements of the members of a filter of subsets. However, this reduction is not available in the context of vector spaces, so in that context, the dual objects are of separate interest. Hence, let us make

Definition 2. *Let X be a right vector space over a division ring k.*

By a linear cofilter *on X we shall mean a nonempty set $\mathcal{G}$ of subspaces of X such that*

(i) *If $A \supseteq B$ are subspaces of X, then $A \in \mathcal{G} \Rightarrow B \in \mathcal{G}$.*

(ii) *If $A, B \in \mathcal{G}$ then $A + B \in \mathcal{G}$.*

A linear endomorphism f of X will be said to respect *the cofilter $\mathcal{G}$ if for all $A \in \mathcal{G}$, $f(A) \in \mathcal{G}$, while f will be called* trivial *with respect to $\mathcal{G}$ if it satisfies the stronger condition $f(X) \in \mathcal{G}$. The ring (or if k is assumed commutative, the k-algebra) of endomorphisms of X which respect $\mathcal{G}$ will be denoted $R(\mathcal{G})$, the ideal of $R(\mathcal{G})$ consisting of endomorphisms trivial with respect to $\mathcal{G}$ will be denoted $I(\mathcal{G})$, and the factor-ring $E(\mathcal{G}) = R(\mathcal{G})/I(\mathcal{G})$ will be called the* eigenring *of* $\mathcal{G}$.

A linear cofilter will be called proper *if it does not contain all subspaces of* X, *equivalently, if it does not contain* X *itself. A maximal proper linear cofilter will be called a* linear co-ultrafilter *on* X.

Miscellaneous remarks. In the above definitions we have spoken of our eigenrings as k-algebras if k is commutative, but merely as rings in the general case. Actually, one can see that for general k, they are algebras over the *center* of k.

Note that for $\mathcal{F}$ a linear filter on X, two elements of $R(\mathcal{F})$ are congruent modulo $I(\mathcal{F})$ if and only if they agree on some member of $\mathcal{F}$, while for $\mathcal{G}$ a linear cofilter, two elements of $R(\mathcal{G})$ are congruent modulo $I(\mathcal{G})$ if and only if the range of their difference is a member of $\mathcal{G}$.

In studying *set-theoretic* filters (especially ultrafilters), one often calls a member of the filter a ''large'' set, and a set whose complement is large ''small''. However, it would be misleading, in considering a linear (ultra)filter $\mathcal{F}$, to think of a linear complement of a member of $\mathcal{F}$ as ''small'', since the sum of two such complements can be a member of $\mathcal{F}$, or, indeed, all of X. (By a *linear complement* of a subspace $A \subseteq X$ we mean a subspace B such that $X = A \oplus B$.) This is an instance of the fact that necessitated our making separate definitions of linear ''filter'' and ''cofilter'': the set of linear complements of members of a linear filter $\mathcal{F}$ does not, in general, constitute a linear cofilter.

How may we think of the rings $E(\mathcal{F})$ and $E(\mathcal{G})$? Loosely, $E(\mathcal{F})$ can be thought of as the ring of endomorphisms of the ''virtual intersection'' of $\mathcal{F}$ (even if the actual intersection is $\{0\}$), and $E(\mathcal{G})$ as the ring of endomorphisms of the factor-space of X by the ''virtual union'' of $\mathcal{G}$. (In §7 we shall construct categories in which such ''virtual intersections'' and ''virtual factor-spaces'' live, and where $E(\mathcal{F})$ and $E(\mathcal{G})$ are indeed their rings of endomorphisms.)

Barbara Osofsky has pointed out to us that our linear filters and cofilters correspond respectively to left and right ideals of the full ring of endomorphisms of X. Indeed, from the easily verified fact that every principal left (respectively, right) ideal of a ring of the form $\mathrm{End}(X)$ is uniquely determined by a subspace of X, namely the common kernel (respectively, the common image) of its cyclic generators, one can deduce that general left and right ideals of $\mathrm{End}(X)$ are in bijective correspondence with filters $\mathcal{F}$, respectively cofilters $\mathcal{G}$ of subspaces of X. For any linear filter $\mathcal{F}$, the corresponding left ideal of $\mathrm{End}(X)$ is, in fact, $I(\mathcal{F})$, and for any linear cofilter $\mathcal{G}$, the corresponding right ideal is $I(\mathcal{G})$. Our $R(\mathcal{F})$ and $R(\mathcal{G})$ are the *idealizer* subrings [**7**, p.30] of these one-sided ideals in $\mathrm{End}(X)$, hence our eigenrings $E(\mathcal{F})$ and $E(\mathcal{G})$ are the endomorphism rings of the left and right $\mathrm{End}(X)$-modules $\mathrm{End}(X)/I(\mathcal{F})$ and $\mathrm{End}(X)/I(\mathcal{G})$ respectively. (One may or may not need to put an ''$(\ )^{\mathrm{op}}$'' on the endomorphism rings of the left modules, depending on one's convention on how such endomorphisms are written. If they are written on the right and composed accordingly, this is not needed.) Simple modules over such rings have been studied using these ideas in [**16**, §§2, 5], [**18**].

However, in what follows we will use the formulations given in the Definitions above, and only occasionally refer to these translations into the language of End(X)-modules. The more general theory that we shall develop in §§7-9 does not seem to admit a comparable translation.

Certain facts about set-theoretic ultrafilters do have analogs for the linear case:

Lemma 3. *Every proper linear filter* (*cofilter*) *on a vector space* X *is contained in a linear ultrafilter* (*co-ultrafilter*).

A linear filter $\mathcal{F}$ (*a linear cofilter* $\mathcal{G}$) *is a linear* (*co*)*ultrafilter if and only if for every subspace* $A \subseteq X$, *either* A *or some linear complement of* A *in* X *belongs to* $\mathcal{F}$ (*respectively,* $\mathcal{G}$).

Proof. The first assertion is clear by Zorn's Lemma.

To see the second, note that a linear filter $\mathcal{F}$ will be a linear ultrafilter if and only if for every subspace A of X, either A is already in $\mathcal{F}$, or the linear filter gotten by adjoining A to $\mathcal{F}$ is improper. That filter consists of all subspaces which contain $A \cap B$ for some $B \in \mathcal{F}$; hence it is improper (contains $\{0\}$) if and only if $\mathcal{F}$ contains some B whose intersection with A is trivial. But every such B is contained in a linear complement to A, and this will also lie in $\mathcal{F}$, giving the desired criterion. The criterion for a linear cofilter to be a linear co-ultrafilter is proved by the dual argument. □

We spoke above of ''adjoining'' an element to a linear filter or cofilter. Let us set up the relevant language and notation.

Definition 4. *Let* S *be any set of subspaces of* X.

Then the least linear filter on X *containing* S, *i.e., the set of all subspaces* $A \subseteq X$ *such that* A *contains a finite intersection of members of* S, *will be called the linear filter* generated *by* S. *If* $\mathcal{F}$ *is any linear filter on* X, *then* $\mathcal{F} \vee S$ *will denote the linear filter generated by* $\mathcal{F} \cup S$, *i.e., the set of all subspaces of* X *that contain the intersection of a member of* $\mathcal{F}$ *and a finite family of members of* S.

Similarly, the least linear cofilter on X *containing* S, *i.e., the set of all subspaces* $A \subseteq X$ *such that* A *is contained in the sum of finitely many members of* S, *will be called the linear cofilter generated by* S, *and if* $\mathcal{G}$ *is a linear cofilter, then* $\mathcal{G} \vee S$ *will denote the linear cofilter generated by* $\mathcal{G} \cup S$, *i.e., the set of all subspaces of* X *that are contained in the sum of a member of* $\mathcal{G}$ *and a finite family of members of* S.

A linear filter or cofilter generated by a single subspace of X *will be called* principal.

Recall that we defined $R(\mathcal{F})$, for $\mathcal{F}$ a linear filter, as the set of linear endomorphisms f of X such that *inverse* images under f of members of $\mathcal{F}$ lie in $\mathcal{F}$, while for $\mathcal{G}$ a linear cofilter, we defined $R(\mathcal{G})$ as the set of f such that *images* of members of $\mathcal{G}$ lie in $\mathcal{G}$. The next result shows that in the cases of linear *ultrafilters* and co-*ultrafilters*, and nontrivial f, we also have the reverse implications.

Lemma 5. *If* U *is a linear ultrafilter on a vector space* X, *and* $f \in R(U) \setminus I(U)$, *then* $A \in U \Rightarrow f(A) \in U$.

Likewise, if V *is a linear co-ultrafilter on* X *and* $f \in R(V) \setminus I(V)$, *then* $A \in V \Rightarrow f^{-1}(A) \in V$.

Proof. Given U, f and A as in the first statement, suppose $f(A) \notin U$. Then by the preceding Lemma, $f(A)$ has a linear complement B in X which belongs to U. Hence $f^{-1}(B)$ belongs to U, hence so does $f^{-1}(B) \cap A$. But this intersection is contained in $\ker(f)$, so $\ker(f) \in U$, i.e., $f \in I(U)$, a contradiction.

The second assertion is proved dually, interchanging the roles of image and inverse image, and of intersection and sum. □

We now arrive at the subject of this paper. The easiest way to get the next result from known facts is to regard $E(U)$, respectively $E(V)$ as endomorphism rings of simple $\mathrm{End}(X)$-modules, and to recall that the endomorphism ring of a simple module is a division ring. Thus, the reader who wishes may skip the proof given below! However, we wanted to give an argument in the spirit of the rest this note, and felt it would provide a good warm-up for arguments of a similar flavor to come.

Proposition 6. *If* U *is a linear ultrafilter, then* $E(U)$ *is a division ring.*

Likewise, if V *is a linear co-ultrafilter, then* $E(V)$ *is a division ring.*

Proof. To get the first assertion, we first note that because U is a *proper* linear filter, the identity endomorphism of X is not in $I(U)$, hence that $1 \neq 0$ in $E(U)$. Now consider any $f \in R(U) \setminus I(U)$. Since $f \notin I(U)$, $\ker(f) \notin U$, hence $\ker(f)$ has a linear complement $A \in U$, and f gives an isomorphism $A \to f(X)$. Let us take the inverse isomorphism, and extend it in an arbitrary way from $f(X)$ to all of X, getting an endomorphism g of X. To show that $g \in R(U)$, note that the inverse image under g of any subspace $B \in U$ contains $f(A \cap B)$. But $A \cap B$ belongs to U, hence by the preceding Lemma, so does $f(A \cap B)$, as required. Finally, we note that fg and gf agree with the identity on the subspaces $f(X)$ and A respectively, of which the latter belongs to U by assumption, and the former by the preceding Lemma. Hence the images in $E(U)$ of fg and gf are 1. Thus, the image of f in $E(U)$ is invertible, showing that $E(U)$ is a division ring.

We sketch the proof of the second assertion. Again, $1 \neq 0$ in $E(V)$ because V is proper. Given $f \in R(V) \setminus I(V)$, we have $f(X) \notin V$, hence $f(X)$ has a linear complement $B \in V$. Letting A be a linear complement of $\ker(f)$ in X, f induces an isomorphism $A \to f(X)$. Taking the inverse to this map, and extending it to all of X by making it zero on B, we get an endomorphism g of X. We find that the composite maps fg and gf differ from the identity map of X by maps having range in $B \in V$ and $\ker(f) = f^{-1}(B) \in V$ respectively; moreover, the former observation shows that for any $C \in V$, $g(C) \subseteq f^{-1}(C+B)$, which by the preceding Lemma lies in V, hence g belongs to $R(V)$, and we conclude that its image in $E(V)$ is an inverse to the image of f. □

There is a degenerate class of cases in which these division rings are easy to characterize. The reader can easily verify

Lemma 7. *The linear filter* (*respectively, cofilter*) *generated by any finite set* S *of subspaces of* X *is principal, generated by the intersection* (*respectively, the sum*) *of the spaces in* S. *The principal linear filter* (*cofilter*) *generated by a subspace* A *is a linear ultrafilter* (*co-ultrafilter*) *if and only if* A *has dimension* 1 (*respectively, codimension* 1). *In this case, its eigenring is naturally isomorphic to the division ring of endomorphisms of* A (*respectively, of* X/A), *and is thus isomorphic to* k (*naturally, or after a choice of basis-element, depending on whether or not* k *is commutative*). $\square$

Let us note

Corollary 8. *A linear ultrafilter* $\mathcal{U}$ *on* X *is nonprincipal if and only if it contains the linear filter of all subspaces of finite codimension in* X.

A linear co-ultrafilter $\mathcal{V}$ *on* X *is nonprincipal if and only if it contains the linear cofilter of all finite-dimensional subspaces of* X.

Proof. In each case, ''if'' is clear from the characterization of the principal linear (co)ultrafilters in the preceding Lemma.

Conversely, suppose $\mathcal{U}$ is a linear ultrafilter and A a subspace of finite codimension in X not belonging to $\mathcal{U}$. Then $\mathcal{U}$ contains a linear complement B to A, which will be finite-dimensional. Since $\mathcal{U}$ is closed under intersections, and the lattice of subspaces of B has descending chain condition, there will be a least subspace of B belonging to $\mathcal{U}$, and this is easily seen to generate $\mathcal{U}$ as a linear filter; thus $\mathcal{U}$ is principal. The assertion for co-ultrafilters is proved similarly. $\square$

We remark that under the correspondence between linear (co)ultrafilters on X and simple $\mathrm{End}(X)$-modules, the *principal* ultrafilters correspond to left modules isomorphic to X itself, and the principal co-ultrafilters to right modules isomorphic to the dual space X^*; both of these simple modules have endomorphism ring k.

What division rings – or more precisely, what division algebras over the center of k – occur as eigenrings of *nonprincipal* linear ultrafilters and co-ultrafilters? Without set-theoretic assumptions, the authors know nothing whatsoever of the answer; even for k commutative, we do not know whether these division algebras all reduce to k, or whether any of them are commutative at all. However, in the next three sections we shall show that if we assume the Continuum Hypothesis, or certain consequences thereof, then both commutative and noncommutative examples occur. Even with such assumptions, we know no *restrictions* on the division algebras that can occur.

Every filter of subsets of a set is an intersection of ultrafilters. The straightforward analog of this statement for vector spaces is false, but we have the elegant result

Proposition 9. *If* $\mathcal{F}$ *is a linear filter* (*respectively,* $\mathcal{G}$ *a linear cofilter*) *on a vector space* X, *then the intersection of all linear ultrafilters containing* $\mathcal{F}$ (*respectively, of all linear co-ultrafilters containing* $\mathcal{G}$) *consists of those subspaces* $A \subseteq X$ *such that every linear complement of* A *has a linear complement belonging to* $\mathcal{F}$ (*respectively, to* $\mathcal{G}$).

Proof. If A has the indicated property, then a linear (co)ultrafilter containing our given (co)filter cannot contain any linear complement of A, hence, by Lemma 3, must contain A.

On the other hand, if A does not have the indicated property, then it has a linear complement B which has no linear complement in our given (co)filter, hence there is no member of that (co)filter whose intersection with B is zero (respectively, whose sum with B is all of X). Hence we can adjoin B to our (co)filter and still get a proper (co)filter, and this will be contained in a (co)ultrafilter. Since this contains B, it does not contain A, showing that A is not in the intersection of all linear (co)ultrafilters containing the given linear (co)filter. □

As an example, we have

Corollary 10. *Let* X *be an infinite-dimensional vector space,* S *a basis of* X, *and* $\mathcal{F}$ *the linear filter consisting of all subspaces of* X *which contain all but finitely many members of* S. *Then the intersection of all linear ultrafilters containing* $\mathcal{F}$ *is the filter of all subspaces of finite codimension in* X.

Proof. This follows immediately from the above Proposition and the fact that every subspace B of X has a linear complement spanned by a subset T of the basis S. (The latter statement may be proved by well-ordering S, and taking for T the set of those elements of S that do not occur as largest element of the ''support'' of any element of B.) □

In the situation of the above Corollary, the intersection of all linear ultrafilters containing $\mathcal{F}$ is strictly larger than $\mathcal{F}$, since, for example, the kernel of the linear functional taking all elements of S to 1 belongs to that intersection, but not to $\mathcal{F}$. We do not know of a corresponding example for linear cofilters. The analogous approach would seem to be to start with the linear cofilter $\mathcal{G}$ of all subspaces contained in the spans of finite subsets of S, and show that the intersection of all linear co-ultrafilters containing this is the linear cofilter of all finite-dimensional subspaces of X. This is true; but in fact, $\mathcal{G}$ itself is the linear cofilter of all finite-dimensional subspaces of X, so this example does not show that our intersection-of-co-ultrafilters construction can change anything!

This asymmetry is a consequence of the fact that an infinite-dimensional vector space has in general ''more'' subspaces of finite codimension than of finite dimension. One can in fact prove that any linear cofilter $\mathcal{G}$ consisting of *finite-dimensional* subspaces of a space X is just the cofilter of all finite-dimensional subspaces of its union, and satisfies the criterion of Proposition 9 for being an intersection of linear co-ultrafilters.

Let us make explicit the meaning of a term we used parenthetically in the proof of Corollary 10, and will be using again.

Definition 11. *Suppose X is a vector space and S a basis for X. Then the* support *of an element $a \in X$ (with respect to the basis S) will mean the set of elements of S occurring with nonzero coefficient in the expression for a as a linear combination of elements of S. The support of a subspace $A \subseteq X$ will mean the union of the supports of the elements of A.*

2. Some lemmas. In the next two sections, we shall develop examples of linear ultrafilters and co-ultrafilters whose eigenrings have specified properties. In this section we will give a few lemmas of general interest that we will need for this task (two known, and one apparently new). On the other hand, lemmas that are most naturally motivated in terms of the specific constructions of the next two sections will be developed in those sections.

Note that we shall in general write ''U'' for a set-theoretic ultrafilter, reserving ''U'' for linear ultrafilters.

Lemma 12 (Katětov [**13**, Proposition 1], cf. [**9**, Theorem 9.2])**.** *Let S be a set, $g\colon S \to S$ a set map, and U a (set-theoretic) ultrafilter on S. Then either the fixed set of g belongs to U, or there is a $T \in U$ such that $g(T)$ is disjoint from T (equivalently, $g^{-1}(T)$ is disjoint from T).*

Sketch of Proof. By an elementary argument (cf. [**13**, Theorem], [**9**, Lemma 9.1]) we can partition S into four subsets such that one of these is the fixed set of g, while each of the others is disjoint from its image under g. One of these sets must belong to U, giving the asserted conclusion. □

Next, we need a bit of terminology. A nonprincipal ultrafilter U on a countable set S is called by specialists a *p-point* if for every partition of S into disjoint subsets $S = S_0 \cup S_1 \cup \dots$, either one of the S_i belongs to U, or there is a member of U that meets each S_i in only finitely many elements, while U is called *q-point* if for every partition of S into disjoint *finite* subsets, $S = S_0 \cup S_1 \cup \dots$, there is a member of U that meets each S_i in at most one element. An ultrafilter U that has both properties is called a *Ramsey ultrafilter*; this can be characterized by the property that for any partition of S into disjoint subsets $S = S_0 \cup S_1 \cup \dots$, either one of the S_i belongs to U, or there is a member of U that meets each S_i in at most one element. For one of the examples in the next section, we will need the existence of a Ramsey ultrafilter, and for another, the weaker assumption that there exists a q-point. Various conditions are known to imply, or be equivalent to, these existence statements; cf. [**1**, §§4.4-4.5]. For the non-specialist reader, we sketch the proof of

Lemma 13 ([**5**, Theorem 4.14], cf. [*ibid.*, Lemma 4.4, Theorem 4.9 (i)⇔(iv)])**.** *The Continuum Hypothesis $2^{\aleph_0} = \aleph_1$ implies the existence of a Ramsey ultrafilter, and hence of a q-point, on any countable set S.*

Sketch of Proof. The set of all partitions of S has cardinality $2^{\aleph_0}$, hence assuming the Continuum Hypothesis, we may index this set as $\{P_\alpha \mid \alpha \in \omega_1\}$, where ω_1 is the first uncountable ordinal. We shall now construct recursively an ascending chain of proper filters F_α $(\alpha \leq \omega_1)$ with the following properties: (i) F_α contains the filter of cofinite subsets of S. (ii) If $\alpha < \omega_1$, then F_α is countably generated. (iii) For each $\beta < \alpha$, F_α either contains a member of the partition P_β, or contains a set which meets each member of P_β in at most one element.

We take F_0 to be the filter of cofinite subsets of S, and at every other limit ordinal α, we let F_α be the union of $\{F_\beta \mid \beta < \alpha\}$; it remains to describe how to construct $F_{\alpha+1}$ from F_α for $\alpha < \omega_1$. Let a countable generating set for F_α be $\{X_0, X_1, \ldots\}$. Then F_α is also generated by the descending chain $X_0 \supseteq (X_0 \cap X_1) \supseteq (X_0 \cap X_1 \cap X_2) \supseteq \ldots$; let us abbreviate this to $Y_0 \supseteq Y_1 \supseteq \ldots$. Let the sets comprising the partition P_α be $S_0, S_1, \ldots$.

If some S_{i_0} has nonempty intersection with all Y_j, choose one such i_0 and define $F_{\alpha+1} = F_\alpha \vee \{S_{i_0}\}$. It is straightforward to verify that this has the desired properties.

In the contrary case, let $F_{\alpha+1} = F_\alpha \vee \{T\}$ where $T = \{t_0, t_1, \ldots\}$ is constructed as follows. Assuming $\{t_0, \ldots, t_{i-1}\}$ chosen, take t_i to be a member of Y_i which does not belong to the same S_j as any of $t_0, \ldots, t_{i-1}$. (This is possible because, by assumption, the set S_j to which each of these finitely many elements belongs has empty intersection with some Y_k; hence the union of all the remaining sets S_j must have nonempty intersection with all Y_k.) Again, it is straightforward that the filter $F_{\alpha+1} = F_\alpha \vee \{T\}$ has the desired properties.

If we now take an ultrafilter U containing F_{ω_1}, condition (iii) guarantees that U will be a Ramsey ultrafilter. □

We remark that *without* any extra set-theoretic assumptions, one can show that there are ultrafilters which are *not* p-points, and ultrafilters which are *not* q-points. To get the former, take a partition of the natural numbers into infinitely many infinite subsets S_i, let F be the filter generated by the complements of all these subsets, together with the complements of all sets which meet each S_i in a finite set, and take any ultrafilter U containing F. To get the latter, let U be any ultrafilter on the natural numbers containing the filter of subsets of ‘‘density 1’’, i.e., subsets S such that $\lim_{n\to\infty} \mathrm{card}(S \cap n)/n = 1$. Then U will contain no sets of density 0. But any set which meets each of the finite sets $\{0\}$, $\{1, 2\}$, $\{3, 4, 5\}$, ... (where the nth set consists of the first n natural numbers not yet used) in at most one element must have density 0; so U is not a q-point. (With a little more work, one can get an ultrafilter which is neither a p-point nor a q-point.) It is also known that the *nonexistence* of a p-point [**1**, Theorem 4.4.7] and the nonexistence of a q-point [**15**] are each consistent with ZFC.

The above two Lemmas will be used in the constructions of §3. For §4 we need a different sort of result:

Lemma 14. *Let* $k \subseteq L$ *be division rings with* $[L:k]_{\mathrm{right}} < \infty$, *let* A *be a right* L*-vector space, and let* B *be a* k*-subspace of* L.

Then a sufficient condition for there to exist an L*-linear functional* $f\colon A \to L$ *such that* $f(B) = L$ *is that* $\dim_k(B) \geq ([L:k]_{\mathrm{right}})^2$. *Likewise, a sufficient condition for there to exist a nonzero element* $a \in A$ *such that* $B \cap aL = \{0\}$ *is that the* k*-codimension of* B *in* A *be at least* $([L:k]_{\mathrm{right}})^2$.

Proof. To get the first assertion, note that a set of $([L:k]_{\mathrm{right}})^2$ k-linearly independent elements of B must include at least $[L:k]_{\mathrm{right}}$ L-linearly independent elements. Hence we can choose an L-linear functional f carrying these onto a right k-basis of L.

In proving the second assertion, let us first show that we can assume A and B finite-dimensional. To do this, we take $([L:k]_{\mathrm{right}})^2$ elements of A whose images in A/B are k-linearly independent, let A_0 be the L-subspace of A that they span, and let $B_0 = B \cap A_0$. Then B_0 still has k-codimension at least $([L:k]_{\mathrm{right}})^2$ in A_0, and we see that if in this finite-dimensional context we can find $a \in A_0$ such that $B_0 \cap aL = \{0\}$, then we in fact have $B \cap aL = \{0\}$, as desired.

So assume A and B finite-dimensional. If B contains at least $[L:k]_{\mathrm{right}}$ L-linearly independent elements, then as in the first paragraph above, we can get an L-linear functional f on A which maps B onto L. Thus, on replacing A and B by $A \cap \ker(f)$ and $B \cap \ker(f)$ respectively, we get a smaller-dimensional L-space and k-subspace, yielding the same k-codimension, and an element a having the indicated property for this pair will have the same property for A and B. Hence by induction we are reduced to the case where the L-subspace $BL \subseteq A$ spanned by B has L-dimension $< [L:k]_{\mathrm{right}}$. However, since B is assumed to have k-codimension at least $([L:k]_{\mathrm{right}})^2$ in A, the k-dimension of A must be at least that large, so its L-dimension must be $\geq [L:k]_{\mathrm{right}} > \dim_L(BL)$. Thus A properly contains BL, and we see that any $a \in A \setminus BL$ will have the desired property. □

(There is clearly a gap between the *sufficient* conditions of the above Lemma, and the obvious *necessary* conditions for the existence of the indicated functional f or element a, namely that B have k-dimension, respectively k-codimension, at least $[L:k]_{\mathrm{right}}$. It is shown in [**3**] that if k and L are fields, or more generally, division algebras finite-dimensional over a common field, then those necessary conditions are also sufficient. For arbitrary division rings, it is shown there that the values $([L:k]_{\mathrm{right}})^2$ of the above Lemma can be reduced by about a factor of 2. We do not know whether one can do any better than that.)

We are now ready to construct some linear ultrafilters and co-ultrafilters.

3. Constructions based on set-theoretic ultrafilters. We will use the following notation.

Definition 15. *Let k be a division ring, X a right k-vector space, and S a k-basis for X. Then for any subset T of S, Tk will denote the span of T in X. If F is a filter of subsets of S, then Fk will denote the linear filter on X generated by the subspaces Tk $(T \in F)$; equivalently, the linear filter of all subspaces of X whose intersection with S belongs to F.*

We shall abbreviate the relative complement $S \setminus T$ of a subset $T \subseteq S$ by T^{c}, and for F a filter on S, we shall write $(F)^{\mathsf{c}}$ for the set of relative complements in S of the members of F (a ''set-theoretic cofilter''). We shall write $(F)^{\mathsf{c}} k$ for the linear cofilter *on X generated by the subspaces $T^{\mathsf{c}} k$ $(T \in F)$; equivalently, the cofilter consisting of all subspaces $A \subseteq X$ whose support in S (the set of basis elements involved in the expressions of the elements of A) is the complement of a member of F.*

If U is an ultrafilter on S, then we shall simplify the symbols $(U)^{\mathsf{c}}$ and $(U)^{\mathsf{c}} k$ to U^{c} and $U^{\mathsf{c}} k$, since in this case, the set of complements (in S) of members of U is precisely the complement of U (in the power set of S).

We shall soon determine just when the linear (co)filters described above, and certain related ones, are linear (co)ultrafilters. In the next Lemma, ignoring this question, we shall compute their eigenrings. The second paragraph of that Lemma refers to the concept of *regular cardinal*; recall that this means an infinite cardinal κ which is not the supremum of a set of $< \kappa$ cardinals smaller than κ; equivalently, an infinite cardinal which, as an ordinal, has no cofinal subset of smaller cardinality. An infinite cardinal that is not regular is called *singular*. (Thus, the least singular cardinal is $\aleph_\omega$.)

Lemma 16. *Let k be a division ring, X a right k-vector space, S a k-basis for X, and U a nonprincipal ultrafilter on S. Then the rings $E(Uk)$ and $E(U^{\mathsf{c}} k)$ are both isomorphic to the ultrapower of k with respect to the ultrafilter U.*

If moreover $\mathrm{card}(S)$ *is regular, and U contains no set of cardinality* $< \mathrm{card}(S)$, *then, denoting by $\mathbf{C}$ the linear filter of all subspaces of X of codimension* $< \mathrm{card}(S)$, *the ring $E(Uk \vee \mathbf{C})$ is also isomorphic to the ultrapower of k with respect to U.*

Proof. Writing k^S for the ring of k-valued functions on S, let us map this into the ring of endomorphisms of the vector space X by taking each $\varphi \in k^S$ to the operator f which carries every $s \in S$ to $s\,\varphi(s)$. It is straightforward that these operators lie in $R(Uk)$, in $R(U^{\mathsf{c}} k)$, and in $R(Uk \vee \mathbf{C})$, and that, moreover, if φ lies in

$$I_U \;=\; \{\, \psi \in k^S \mid \{ s \in S \mid \psi(s) = 0 \} \in U \},$$

then f lies in $I(Uk)$, $I(U^{\mathsf{c}} k)$, and $I(Uk \vee \mathbf{C})$. Hence we have natural homomorphisms from k^S / I_U to the rings $E(Uk)$, $E(U^{\mathsf{c}} k)$ and $E(Uk \vee \mathbf{C})$. Since k^S / I_U is the ultrapower of k with respect to U, and is thus a division ring, these homomorphisms are one-to-one. (To get one-one-ness in the case of $E(Uk\vee\mathbf{C})$, we need the hypothesis on cardinalities of members of U to conclude that $E(Uk\vee\mathbf{C})\neq 0$.) It remains only to show that these embeddings are surjective.

To show this for the embedding in $E(Uk)$, consider any $f\in R(Uk)$. Define a map $g\colon S\to S$ to fix all $s\in S$ which are ''eigenvectors'' of f (i.e., are sent by f to an element of the form $s\varphi(s)$), and to send every other element s to some element $t\neq s$ in the support of $f(s)$ with respect to the basis S. Now by Lemma 12, either the fixed set of g is in U, or U contains a set T whose image under g is disjoint from T. If the latter is true, note that by continuity of f we have $f^{-1}(Tk)\in Uk$, hence $f^{-1}(Tk)\cap Tk\in Uk$; yet by assumption on T, this intersection contains no member of S, a contradiction. Hence the fixed set of g lies in U. Now calling that set T, we see that on Tk, f may be described as multiplying each $t\in T$ by an element $\varphi(t)\in k$. If we extend φ in an arbitrary manner to a map $S\to k$, we see that the endomorphism of X which sends s to $s\varphi(s)$ for all $s\in S$ agrees with f on $Tk\in Uk$, hence the image of f in $E(Uk)$ is in the image of k^S/I_U, as claimed.

The proof for $E(U^{\mathbf{c}}k)$ is similar. Given $f\in R(U^{\mathbf{c}}k)$, let $g\colon S\to S$ be a map which fixes $s\in S$ if and only if s does not occur in the support of $f(t)$ for any $t\in S\setminus\{s\}$, and otherwise sends s to some $t\neq s$ such that s lies in the support of $f(t)$. As before, either the fixed set of g is a member of U, or there is some $T\in U$ whose image under g is disjoint from T. In the latter case, we note that $T^{\mathbf{c}}k\in U^{\mathbf{c}}k$, but that the image of this subspace under f involves every member of T in its support, hence is not in $U^{\mathbf{c}}k$, a contradiction. Hence the fixed set of g belongs to U, and now calling this T, we see that for each $s\in S$, $f(s)$ agrees modulo $T^{\mathbf{c}}k$ with a scalar multiple of s, so again the image of f in our eigenring lies in the image of k^S/I_U.

The case of $E(Uk\vee\mathbf{C})$ is trickier. Let us choose an ordering of S of the order type of $\mathrm{card}(S)=\dim(X)$. Thus, every proper initial segment of S has cardinality $<\mathrm{card}(S)$, and conversely (by our assumption that $\mathrm{card}(S)$ is regular), every subset of S of cardinality $<\mathrm{card}(S)$ lies in a proper initial segment. Now given $f\in R(Uk\vee\mathbf{C})$, define $g\colon S\to S$ to take each $s\in S$ to itself if it is an eigenvector of f, otherwise to the *largest* member of the support of $f(s)$ other than s itself, under this ordering. Again, by Lemma 12 there exists a $T\in U$ such that either T consists of fixed points of g, or $g(T)$ is disjoint from T. If T consists of fixed points, i.e., eigenvectors of f, the desired conclusion follows as in the first case, so assume the contrary. Since $T\in U$, we have $Tk\in Uk\vee\mathbf{C}$, hence $f^{-1}(Tk)\in Uk\vee\mathbf{C}$. This means that there exists $T'\in U$ such that some subspace of $T'k$ of codimension $<\mathrm{card}(S)$ therein is contained in $f^{-1}(Tk)$. Hence some subspace of $f(T'k)$ of codimension $<\mathrm{card}(S)$ is contained in Tk; equivalently, there exists $Q\subseteq S$ of cardinality $<\mathrm{card}(S)$ such that $f(T'k)\subseteq(T\cup Q)k$. By the properties of our ordering of S, we may take Q to be an initial segment of S.

Now consider the behavior of f on an element $t\in T\cap T'$. Because $t\in T$, the largest member of the support of $f(t)$, other than possibly t itself, is $g(t)$, which is not a member of T; but because $t\in T'$, the latter element must be a member of

$T \cup Q$; hence it must be a member of Q. Furthermore, since Q was assumed an initial segment of S, from the fact that that element belongs to Q it follows that *all* members of the support of $f(t)$ other than t belong to Q. This says that the restriction of f to $(T \cap T')k \in Uk$ is the sum of an operator h taking each member of $T \cap T'$ to a scalar multiple of itself, and an operator q with range in Qk. Since Q has cardinality $<\mathrm{card}(S)$, we have $\ker q \in \mathbf{C}$, so $q \in I(Uk \vee \mathbf{C})$. Hence the image of f in $E(Uk \vee \mathbf{C})$ is the same as that of h, so f has the required form. □

The first assertion of the above Lemma (concerning $E(Uk)$) has been obtained independently by N. Rodinò [**19**].

We shall use the above Lemma only in the case where S is countable; but comparing the assertions of that Lemma regarding $E(Uk)$ and $E(Uk \vee \mathbf{C})$, it is natural to ask whether, if S is uncountable and κ is an infinite cardinal $<\mathrm{card}(S)$, the last assertion of the Lemma remains true with $\mathbf{C}$ replaced by the smaller filter $\mathbf{C}_\kappa$ of subspaces of X of codimension $<\kappa$. The above proof does not go over to this situation, because the step where we enlarged Q to the initial segment of S that it generated, though it preserved the property that Q had cardinality $<\mathrm{card}(S)$, does not preserve the property of having cardinality $<\kappa$. However, the generalized result can be proved from the $E(Uk \vee \mathbf{C})$ case:

Corollary 17. *Suppose k, X, S and U are as in the last paragraph of the preceding Lemma, let κ be any cardinal $\leq \mathrm{card}(S)$, and let $\mathbf{C}_\kappa$ denote the linear filter of all subspaces of X of codimension $<\kappa$. Then $E(Uk \vee \mathbf{C}_\kappa)$ is isomorphic to the ultrapower of k with respect to the ultrafilter U.*

Sketch of Proof. As before, it suffices to prove that any member of $R(Uk \vee \mathbf{C}_\kappa)$ is the sum of a ''diagonal'' element and an element of $I(Uk \vee \mathbf{C}_\kappa)$. Now it is easy to see that *every* endomorphism of X is continuous with respect to the linear filter $\mathbf{C}$ defined in the preceding Lemma, and from this it is easily deduced that $R(Uk \vee \mathbf{C}_\kappa) \subseteq R(Uk \vee \mathbf{C})$. Hence applying that Lemma, we see that every member of $R(Uk \vee \mathbf{C}_\kappa)$ is the sum of a ''diagonal'' element and an element

$$f \in I(Uk \vee \mathbf{C}) \cap R(Uk \vee \mathbf{C}_\kappa).$$

So it suffices to prove that every such f lies in $I(Uk \vee \mathbf{C}_\kappa)$.

Note that the condition $f \in I(Uk \vee \mathbf{C})$ is equivalent to saying that there exists $T \in U$ such that $f(T) \subseteq Qk$ for some Q of cardinality $<\mathrm{card}(S)$. (Cf. argument where ''Q'' first appears in the proof of the preceding Lemma. That concerned membership in $R(Uk \vee \mathbf{C})$; this is the analogous observation on $I(Uk \vee \mathbf{C})$.) By our assumption on U, we have $Q^{\mathsf{c}} \in U$, so as $f \in R(Uk \vee \mathbf{C}_\kappa)$, we have $f^{-1}(Q^{\mathsf{c}} k) \in Uk \vee \mathbf{C}_\kappa$; hence $f^{-1}(Q^{\mathsf{c}} k) \cap Tk \in Uk \vee \mathbf{C}_\kappa$. But by choice of Q, the image of this space under f is zero, showing that $f \in I(Uk \vee \mathbf{C}_\kappa)$, as desired. □

We do not know whether any version of the above Corollary can be proved when $\mathrm{card}(S)$ is a singular cardinal.

Note that in the context of Lemma 16, Uk is strictly smaller than $Uk \vee \mathbf{C}$, hence Uk is not a linear ultrafilter. Thus, the $E(Uk)$ case of that Lemma shows that there are linear filters other than linear ultrafilters whose eigenrings are division rings.

On the other hand, the next result will show that assuming the existence of q-points, respectively Ramsey ultrafilters, some of the linear filters and cofilters described in the other two cases of that Lemma *are* linear (co)ultrafilters. Thus, we will have examples of nonprincipal linear ultrafilters and co-ultrafilters whose eigenrings we can describe fairly exactly.

Proposition 18. *Let k be a division ring, X a countable-dimensional right k-vector-space, with basis S, $\mathbf{C}$ the filter of all subspaces having finite codimension in X, and U a nonprincipal ultrafilter on S. Then the linear filter $Uk \vee \mathbf{C}$ is a linear ultrafilter if and only if U is a Ramsey ultrafilter, while the linear cofilter $U^{\mathrm{c}}k$ is a linear co-ultrafilter if and only if U is a q-point.*

Proof. First suppose U is a Ramsey ultrafilter, and A any subspace of X. We must prove that either A or some linear complement of A belongs to $Uk \vee \mathbf{C}$. Let us write X as the union of a chain of subspaces (not necessarily all distinct), each of finite codimension in the next, beginning with A:

(1) $$A \;=\; A_0 \;\subseteq\; A_1 \;\subseteq\; A_2 \;\subseteq\; \dots\,.$$

Let S_0 denote the set of elements of S that lie in $A = A_0$, and for $i > 0$, let S_i denote the set of elements of S that lie in A_i but not in A_{i-1}. Clearly, the sets S_i partition S. If some S_i is a member of U, then we have a term of (1) which is a member of Uk; hence A, which has finite codimension in that space, belongs to $Uk \vee \mathbf{C}$. In the contrary case, since U is Ramsey we can find a $T \in U$ which meets each S_i in at most one element. If T contains an element of $S_0 = S \cap A$, let us drop this element from T; the resulting set clearly still belongs to U. We now claim that Tk has trivial intersection with A. For suppose $0 \neq a \in Tk \cap A$, and let the support of a be $\{t_0, \dots, t_r\} \subseteq T$, with $t_i \in S_{m(i)}$ and $0 < m(0) < \dots < m(r)$. Then $A_{m(r-1)}$ contains $t_0, \dots, t_{r-1}$, and also contains $a \in A = A_0$, hence taking a linear combination of these, it contains t_r, which is impossible, because by construction, members of $S_{m(r)}$ do not appear in any space A_i with $i < m(r)$. Since Tk thus has trivial intersection with A, it can be enlarged to a linear complement of A, which by choice of T will belong to $Uk \subseteq Uk \vee \mathbf{C}$, as required.

Conversely, suppose the linear filter $Uk \vee \mathbf{C}$ is a linear ultrafilter, and let P be a partition of S. To each $Q \in P$ let us associate the linear functional $f_Q : X \to k$ taking members of Q to 1, and all other members of S to 0; and let $A_P \subseteq X$ be the intersection over all $Q \in P$ of the kernels of these functionals, i.e., the set of elements of X such that the sum of the coefficients of the elements in each member of our partition is 0. By assumption, either (a) A_P belongs to $Uk \vee \mathbf{C}$, or (b) it has a linear complement in $Uk \vee \mathbf{C}$.

Now (a) implies that we can find finitely many elements $s_1, \dots, s_n \in S$ such that $A_P + \{s_1, \dots, s_n\}\, k \in Uk$, i.e., such that the set of elements of S lying in

$A_P + \{s_1, \dots, s_n\}\,k$ is a member of U. But from the definition of A_P, we see that for any $s \in S$, the above space contains s if and only if one of $s_1, \dots, s_n \in S$ lies in the same set $Q \in P$ to which s belongs. Thus, U contains the union of finitely many members of P, so as U is an ultrafilter, one of these belongs to U.

On the other hand, alternative (b) says that there is some $T \in U$ such that A_P has a linear complement which is of finite codimension in a space containing Tk. This is equivalent to saying that $A_P \cap Tk$ is finite-dimensional. Now if T contains n elements of some $Q \in P$, then from the definition of A_P we see that this contributes a subspace of dimension $n-1$ to the above intersection; hence T can meet only finitely many members of P in more than one element, and each of those in only finitely many elements. Since U is a nonprincipal ultrafilter, we can discard finitely many elements, and get a set $T' \in U$ which meets every member of P in at most one element.

Thus, the alternatives (a) and (b) lead to the alternatives which define a Ramsey ultrafilter.

Let us now prove the criterion for $U^{\mathrm{c}}k$ to be a linear co-ultrafilter. First, assume U is a q-point. Given an arbitrary subspace $A \subseteq X$, let us write it as the union of a chain of finite-dimensional subspaces (not necessarily all distinct):

$$\{0\} \;=\; A_0 \;\subseteq\; A_1 \;\subseteq\; A_2 \;\subseteq\; \dots\,.$$

For each i, let S_i denote the finite set of elements of S occurring in the support of A_i but not in the supports of earlier terms of the series. If the union of the S_i, i.e., the support of A, does not lie in U, then A itself belongs to $U^{\mathrm{c}}k$. In the contrary case, it follows from the assumption that U is a q-point that we can find a subset T of the support of A which belongs to U and meets each S_i in at most one element. We claim that $T^{\mathrm{c}}k + A = X$. Indeed, since T^{c} contains all members of S not in any of the S_i, it suffices to verify that for $i > 0$, $T^{\mathrm{c}}k + A$ contains every $s \in S_i$. Assuming inductively that it contains all members of $S_0, \dots, S_{i-1}$, and recalling that T^{c} contains all but at most one member of S_i, we see that if $T^{\mathrm{c}}k + A$ does not contain s it contains all other members of the support of A_i; but now taking any member of A_i which involves the basis element s, and subtracting a linear combination of the other elements of the support of A_i, we get a nonzero scalar multiple of s, a contradiction. So $T^{\mathrm{c}}k + A = X$, so some subspace of $T^{\mathrm{c}}k \in U^{\mathrm{c}}k$ is a linear complement of A, as required.

Finally, suppose $U^{\mathrm{c}}k$ is a linear co-ultrafilter, and let P be any partition of S into finite sets. For each $Q \in P$, let x_Q denote the sum in X of the members of Q, and let A_P be the space spanned by all these elements. (Thus, A_P is the space of those $x \in X$ such that for each $Q \in P$, all members of Q have the same coefficient in x.) Now the support of A_P is all of S, so A_P is not contained in Tk for any proper subset $T \subseteq S$, in particular, for any $T \in U^{\mathrm{c}}$, so $A_P \notin U^{\mathrm{c}}k$. Hence $U^{\mathrm{c}}k$ must contain a linear complement of A_P; i.e., there must be some $T \in U$ such that $A_P + T^{\mathrm{c}}k = X$. But for each $Q \in P$, the condition $A_P + T^{\mathrm{c}}k \supseteq Q$ implies that T^{c} contains all but at most one element of Q. Hence $T \in U$ meets each $Q \in P$ in at most one element, proving that U is a q-point, and completing the proof of our Proposition. □

Combining Lemma 16 and Proposition 18, we get

Theorem 19. *Assuming the Continuum Hypothesis* (*or the weaker statement that there exists a Ramsey ultrafilter, respectively, a q-point*), *a countable-dimensional vector space* X *over a division ring* k *has a linear ultrafilter* $\mathcal{U}$ *and a linear co-ultrafilter* $\mathcal{V}$ *such that* $E(\mathcal{U})$ *and* $E(\mathcal{V})$ *are isomorphic to proper ultrapowers of the division ring* k.

In particular, if k *is commutative,* $E(\mathcal{U})$ *and* $E(\mathcal{V})$ *are commutative.* □

Remark: In our formulation of the constructions given above, we started with a vector space X and took a basis S thereof; but we could equally well have started with a set S, and formed the vector space $X = Sk$ with this as basis. This viewpoint suggests several variant constructions that might be worth studying. First, given a set S and a nonprincipal ultrafilter U on S, suppose we form the right k-vector-space k^S of all k-valued functions on S, and consider the linear filter $\mathcal{U}$ generated by the subspaces k^T with $T \in U$ (where we identify k^T with the space of elements of k^S with support in T). Will this, or some filter obtained from it in a natural way, ever be a linear ultrafilter, and if so what properties will its eigenring have? The latter question, at least, seems hard to attack, because the vector space k^S does not in general have a basis that can be described explicitly in terms of S, so it is difficult to describe its general endomorphism. A (slightly) more tractable construction might be the subspace of k^S which we will temporarily call $k^{[S]} \subseteq k^S$, consisting of those functions on S that assume only finitely many distinct values in k. (The method of [**2**] leads to a description of a basis of this space, given a well-ordering of S.) Finally, we can look at factor-spaces such as $k^S / k^{[S]}$, and $k^{[S]} / Sk$; each of these also has a linear filter induced by U and a linear cofilter induced by U^{c}. Whether these, or any (co)filters obtained from them, will under some conditions be linear ultrafilters we do not know.

Let us end this section with an example of a different sort. We have seen that the rings $E(\mathcal{F})$ are division rings if $\mathcal{F}$ is a linear ultrafilter, and in some cases where it is not (Lemma 16, first paragraph). On the other hand, if $\mathcal{F} = \{X\}$, the least filter on X, then $E(\mathcal{F})$ is the full endomorphism ring of X. The next example shows that a certain pleasant property shared by all these examples is not true of all eigenrings $E(\mathcal{F})$ or $E(\mathcal{G})$.

Example 20. *A linear filter* $\mathcal{F}$ *and a linear cofilter* $\mathcal{G}$ *on a vector space* X *such that* $E(\mathcal{F})$ *and* $E(\mathcal{G})$ *are not von Neumann regular.*

Construction. Let X be a vector space with a basis $S = \{s_{i,j}\}$ indexed by pairs of natural numbers, and F the filter of subsets of S generated by the subsets

$$T_m = \{s_{i,j} \mid j \geq m\}.$$

Let $g\colon S \to S$ be defined by

$$g(s_{i,j}) = s_{i,j+i}.$$

This map is continuous with respect to the filter F, hence the linear map f on X that it induces is continuous with respect to the linear filter Fk; moreover, since f is one-to-one, its image in $E(Fk)$ is a left non-zero-divisor. If $E(Fk)$ were von Neumann regular, this would imply that that element had a left inverse, i.e., that there was an element $h \in R(Fk)$ such that hf agreed with the identity on a member of Fk. Now this says that for all sufficiently large j, one would have $h(s_{i,j+i}) = s_{i,j}$ for all i. But taking such a j, we see that $h^{-1}(T_{j+1}k)$ would not contain $s_{i,j+i}$ for any i, hence would not contain $T_{j+i}k$ for any i, contradicting the continuity of h.

The construction for cofilters is similar. Briefly, take the same X and F, and define the linear map $f'\colon X \to X$ by letting $f'(s_{i,j}) = s_{i,j-i}$ if $j \geq i$, 0 otherwise. This map lies in $R((F)^{\mathrm{c}}k)$, and because it is surjective, its image in $E((F)^{\mathrm{c}}k)$ is a right non-zero-divisor. If this image had a right inverse, induced by some $h' \in R((F)^{\mathrm{c}}k)$, then for all sufficiently large j, the element $h'(s_{i,j})$ would have to have $s_{i,j+i}$ in its support for each i. Thus, h' could not carry $(T_{j+1}{}^{\mathrm{c}})k$ into $(T_{j+i}{}^{\mathrm{c}})k$ for any i. □

4. Eigenrings containing specified extensions of k. We will now obtain some examples contrasting with those of Theorem 19; these will show in particular that if we assume the Continuum Hypothesis, then for k commutative, division rings of the form $E(U)$ and $E(V)$ can be noncommutative. We shall get these examples by taking a countable dimensional vector space X over a finite-dimensional division algebra L over k, regarding X as a k-vector-space, and constructing a k-linear (co)ultrafilter U (respectively, V) on it such that the actions of all elements of L lie in $R(U)$ (respectively, $R(V)$), and hence have images in $E(U)$ (respectively, $E(V)$). Observe that when we regard these (right) multiplication operators as k-linear endomorphisms of X, and so write them on the *left* and compose them accordingly, the subalgebra of $\mathrm{End}_k(X)$ that they comprise is isomorphic to L^{op}, the opposite of the algebra L. Thus, our construction will yield division algebras $E(U)$ and $E(V)$ having subalgebras isomorphic to L^{op}.

More generally, suppose k is a division ring, and L a division ring containing it. In this case, noncentral members of k, and more generally, elements of L that are not in the centralizer of k do not in a natural way induce k-linear endomorphisms of an L-vector-space X; so what the above technique will show, under appropriate hypotheses, is that $E(U)$ and $E(V)$ can be taken to contain copies of the opposite of the *centralizer* D of k in L. Observe that no elements of L other than elements of k and of D come into this statement. Thus, in this situation, we may as well replace L by the division ring generated by those two division subrings, in other words, assume that L is generated by k and D. Now under this assumption, observe that if one of $[L:k]_{\mathrm{right}}$ and $[L:k]_{\mathrm{left}}$ is finite, then L has a k-basis of k-centralizing elements, so these right and left degrees are equal, and we can abbreviate this finiteness condition by simply writing $[L:k] < \infty$. With the above points understood, the following observations help to clarify our task.

Lemma 21. *Let* $k \subseteq L$ *be division rings such that* L *is generated by* k *and the centralizer* D *of* k *in* L, *and such that* $[L:k] < \infty$. *Let* X *be a right* L*-vector-space.*

Then for any k*-linear ultrafilter* $\mathcal{U}$ *on* X, *the following conditions are equivalent:* (i) *The actions of multiplication by all elements of* D *lie in* $R(\mathcal{U})$. (ii) *Every member of* $\mathcal{U}$ *contains a nonzero* L*-subspace of* X. (iii) *For every* k*-subspace* $C \subseteq X$ *which contains no nonzero* L*-subspace of* X, *there is an* $A \in \mathcal{U}$ *such that* $A \cap C = \{0\}$.

Likewise, for any k*-linear co-ultrafilter* $\mathcal{V}$ *on* X, *the following conditions are equivalent:* (i′) *The actions of multiplication by all elements of* D *lie in* $R(\mathcal{V})$. (ii′) *Every member of* $\mathcal{V}$ *is contained in a proper* L*-subspace of* X. (iii′) *For every* k*-subspace* $C \subseteq X$ *which spans* X *as an* L*-vector-space, there is an* $A \in \mathcal{V}$ *such that* $A + C = X$.

Proof. The hypotheses on L imply that we can find a finite set of elements $d_1, \dots, d_n$ of $D \setminus \{0\}$ which span L over k. Thus, given a k-subspace $C \subseteq X$, the L-subspace that it spans is $\Sigma\, C d_i$, and the largest L-subspace it contains is $\cap\, d_i^{-1}(C)$, where in the latter expression, each d_i is being regarded as a k-linear operator on X.

From this last fact we immediately get (i)$\Rightarrow$(ii). Conversely, assuming (ii), we see that the subspaces $\cap\, d_i^{-1}(A)$ for $A \in \mathcal{U}$ will generate a proper linear filter containing $\mathcal{U}$, hence this filter must equal $\mathcal{U}$, which implies (i). The equivalence of (i′) and (ii′) is gotten in the same way.

The equivalences (ii)$\Leftrightarrow$(iii) and (ii′)$\Leftrightarrow$(iii′) follow from the fact that a (co)ultrafilter fails to contain C if and only if it contains a linear complement to C. $\square$

Thus, to prove the existence of a linear (co)ultrafilter with the above equivalent properties, it will suffice to construct a linear (co)filter $\mathcal{F}$ or $\mathcal{G}$ satisfying condition (iii), respectively (iii′), since this condition is preserved under enlarging $\mathcal{F}$ or $\mathcal{G}$. We will do this, under appropriate hypotheses, by a double recursion, with one recursion indexed by $\aleph_0$ and the other by $\aleph_1$. Let us start with the former.

Lemma 22. *Let* k, L *and* X *be as in the first paragraph of the preceding Lemma.*

Suppose $\mathcal{F}$ *is a countably generated filter of infinite-dimensional* L*-subspaces of* X, *and* C *a* k*-subspace of* X *containing no nontrivial* L*-subpace. Then there exists an* L*-subspace* $B \subseteq X$ *such that* $B \cap C = \{0\}$, *and such that every member of the* L*-linear filter* $\mathcal{F} \vee \{B\}$ *is still infinite-dimensional.*

Likewise, assuming X *countable-dimensional, if* $\mathcal{G}$ *is a countably generated cofilter of* L*-subspaces of* X *all members of which have infinite codimension in* X, *and* C *a* k*-subspace of* X *which spans* X *as an* L*-vector-space, then there exists an* L*-subspace* $B \subseteq X$ *such that* $B + C = X$, *and such that every member of the* L*-linear cofilter* $\mathcal{G} \vee \{B\}$ *still has infinite codimension in* X.

Proof. To get the first assertion, note that since $\mathcal{F}$ is countably generated, it is generated by a descending chain indexed by the natural numbers, $A_0 \supseteq A_1 \supseteq \dots$ (cf. end of second paragraph of proof of Lemma 13). We claim we can find elements $b_0 \in A_0$, $b_1 \in A_1$, ... such that no nontrivial L-linear combination of these elements lies in C. Indeed, assume inductively that $b_0, \dots, b_{i-1}$ have been so chosen, and let us write B_{i-1} for the L-subspace of X which they span. Now note that if $A_i \cap C$ had finite codimension in A_i, it would contain an L-subspace also of finite codimension therein (cf. first paragraph of proof of the preceding Lemma). But since C contains no nontrivial L-subspace, and A_i is infinite-dimensional, this is not so, i.e., $A_i \cap C$ has infinite codimension in A_i. Hence $(A_i \cap C) + B_{i-1}$ has infinite codimension in $A_i + B_{i-1}$. Intersecting with A_i (which has finite codimension in the latter space) and using the modular law, we conclude that $A_i \cap (C + B_{i-1})$ has infinite codimension in A_i. Hence by Lemma 14 there exists a nonzero element $b_i \in A_i$ such that $b_i L$ has trivial intersection with $A_i \cap (C + B_{i-1})$, equivalently, has trivial intersection with $C + B_{i-1}$. This is the property required for our recursive construction of the b_i's. The space B spanned by all these elements is easily seen to have the desired properties.

The proof of the second assertion will be similar, but there is an additional complication. In the above argument, we ''built B up'' step by step; in this dual situation, we will obtain it as an intersection of a descending chain. However, though it is clear that if each member of an ascending chain has trivial intersection with a subspace C, so will the union, it is *not* true that if each member of a descending chain has the property that its sum with C is all of X, the same will be true of the intersection. Hence, to guarantee this property, we will have to select, at successive stages of our construction, elements b_i which we will require never to be dropped at subsequent stages, and which together will span X over C. It is to make this possible that we require X to be countable-dimensional.

Let us begin. Since the cofilter $\mathcal{G}$ is countably generated, it has a generating chain $A_0 \subseteq A_1 \subseteq \dots$. Let $S = \{s_0, s_1, \dots\}$ be a basis for X over L. We shall now construct a sequence of L-linear functionals $f_i : X \to L$, and a sequence of elements $b_i \in X$, such that for all i and j, the following conditions hold:

(2) $$b_i - s_i \in C.$$

(3) $$f_i(A_i) \;=\; 0.$$

(4) $$f_j(b_i) \;=\; 0.$$

(5) The k-linear map $(f_0, \dots, f_i) : C \to L^{i+1}$ is surjective.

Assume inductively that $f_0, \dots, f_{i-1}$ and $b_0, \dots, b_{i-1}$ have been found such that all cases of (2)-(5) involving only those elements hold. Then the ''i–1'' case of (5) implies that there exists an element $c_i \in C$ such that

$$(f_0, \dots, f_{i-1})(c_i) \;=\; -(f_0, \dots, f_{i-1})(s_i).$$

Let $b_i = c_i + s_i$. Thus we have condition (2) for this value of i, and (4) for this value of i and all $j < i$. We now wish to choose f_i so as to satisfy all cases of (3), (4) and (5) involving f_i and the b's selected so far. This is equivalent to finding an L-linear functional on $X/(A_i + \{b_0, \dots, b_i\} L)$ which, when restricted to the image in that space of the kernel of the map

$$(f_0, \dots, f_{i-1}) : C \to L^i, \tag{6}$$

is a surjective map to L. Now by Lemma 14, this will certainly be possible if the kernel of (6) has infinite-dimensional image in $X/(A_i + \{b_0, \dots, b_i\} L)$, and this in turn will be true if C has infinite-dimensional image in X/A_i. But since C spans X as an L-vector-space, its image in X/A_i spans that space (which by hypothesis is infinite-dimensional) as an L-vector-space, and it could not do this if it were finite-dimensional over k. This completes our recursive construction of the f_i and b_i.

Now let B be the intersection of the kernels of the f_i; thus, by (4), B contains all b_i, so by (2), $B + C = X$. By (5), the functionals f_i are left L-linearly independent, hence, as each of the spaces A_i is contained in all but finitely many of the others, and so is annihilated by infinitely many of the f_i, the latter is also true of the spaces $A_i + B$, so these have infinite codimension in X. Thus, $\mathcal{F} \vee \{B\}$ is a cofilter of spaces of infinite codimension, completing the proof of the Lemma. □

The recursion over $\aleph_1$ is now easy!

Corollary 23. *Let k, L and X be as in the preceding two Lemmas.*

Suppose $\mathcal{F}_0$ is a countably generated L-linear filter on X such that every member of $\mathcal{F}_0$ is infinite-dimensional, and $\{C_\beta \mid \beta \in \omega_1\}$ is a family of $\leq \aleph_1$ k-subspaces of X, none of which contains a nontrivial L-subpace. Then $\mathcal{F}_0$ is contained in a nonprincipal k-linear ultrafilter $\mathcal{U}$ which contains none of the C_β.

Likewise, assuming X countable-dimensional, if $\mathcal{G}_0$ is a countably generated L-linear cofilter on X such that every member of $\mathcal{G}_0$ has infinite codimension in X, and $\{C_\beta \mid \beta \in \omega_1\}$ a family of $\leq \aleph_1$ k-subspaces of X, each of which spans X over L, then $\mathcal{G}_0$ is contained in a nonprincipal k-linear co-ultrafilter $\mathcal{V}$ which contains none of the C_β.

Proof. In the first case, a transfinite recursive application of the preceding Lemma indexed by ω_1, the first uncountable ordinal, yields an L-linear filter which consists of infinite-dimensional subspaces of X, and which for every $\beta \in \omega_1$ contains a space having zero intersection with C_β. If we adjoin to this filter all L-subspaces of finite codimension in X, and extend the resulting proper L-linear filter to a k-linear ultrafilter $\mathcal{U}$, we see that $\mathcal{U}$ will be nonprincipal, and cannot contain any of the C_β. The second statement is deduced in the same way. □

We can now deduce the main result of this section.

Theorem 24. *Let $k \subseteq L$ be countable division rings such that L is generated over k by the centralizer of k in L, which we shall denote D, and such that $[L:k] < \infty$. Let X be a countably infinite-dimensional right k-vector-space.*

Then assuming the Continuum Hypothesis, there exist a k-linear ultrafilter $\mathcal{U}$ and a k-linear co-ultrafilter $\mathcal{V}$ on X such that the division rings $E(\mathcal{U})$ and $E(\mathcal{V})$ have division subrings isomorphic to D^{op}.

In fact, under the above hypotheses, given any right L-vector-space structure on X extending the k-vector-space structure, and any countably generated L-linear filter $\mathcal{F}_0$ on X consisting of infinite-dimensional subspaces, there exists a k-linear ultrafilter $\mathcal{U}$ which contains $\mathcal{F}_0$, and such that the actions of all members of D on X are continuous with respect to $\mathcal{U}$, thus yielding an embedding $D^{\mathrm{op}} \to E(\mathcal{U})$; and the analogous statement holds for cofilters.

Proof. We shall prove the assertion of the final paragraph.

Since k is countable, so is X, hence it has at most $2^{\aleph_0}$ subsets, in particular at most $2^{\aleph_0}$ k-subspaces, hence assuming the Continuum Hypothesis, at most $\aleph_1$ such subspaces. Hence we can index by $\aleph_1$ the set of those k-subspaces of X which contain no nonzero L-subspace, and apply Corollary 23. An application of Lemma 21 now gives the desired result. The ‘‘cofilters’’ case is exactly parallel. □

One may ask whether the technique used above doesn’t prove too much. For instance, if k is a field, and L a finite-dimensional simple k-algebra that is *not* a division algebra, what is to stop us from arguing, in the same way, that there must be a k-linear ultrafilter $\mathcal{U}$ and a k-linear co-ultrafilter $\mathcal{V}$ with $E(\mathcal{U})$ and $E(\mathcal{V})$ having subalgebras isomorphic to L^{op} – though we know by Proposition 6 that this is impossible? The answer is that Lemma 14 is only true for L a division ring. To see how it fails when L is the 2×2 matrix ring $M_2(k)$, let X_0 be any infinite-dimensional k-vector-space, and let $X = X_0 \oplus X_0$, regarded as consisting of length-2 row vectors of elements of X, on which we let $M_2(k)$ act on the right in the obvious manner. Then the k-subspace $C = X_0 \oplus \{0\}$ is infinite-dimensional, but its image under every right L-module homomorphism $X \to L$ consists of matrices with second column zero. Likewise, it has infinite k-codimension in X, but has nonzero intersection with every nonzero L-submodule of X.

One might also ask whether our approach of turning the right D-vector-space structure of our right L-vector space X into an action of D^{op} by k-vector-space endomorphisms isn’t unnecessarily roundabout. Couldn’t we have started by representing X as a direct sum of copies of L, then found a k-linear (co)ultrafilter with respect to which *left* multiplication by all members of L was continuous, thus getting examples with $L \subseteq E(\mathcal{U})$? The difficulty here is that the combination of left and right L-vector-space structures would be equivalent to an $L^{\mathrm{op}} \otimes_{k_0} L$-module structure on X, where k_0 is the center of L, and since $L^{\mathrm{op}} \otimes_{k_0} L$ is not in general a division ring, the linear algebra involved would have been more complicated. Indeed, since such tensor product algebras are typically matrix rings over division rings, it appears that such an approach would run into the difficulties discussed in the preceding paragraph. Thus, if a construction along these lines is possible, it probably has to be rather different from the one we gave.

On another topic, recall that the proof of our Theorem involved moving between L-linear and k-linear (co)filters on X. Let us look at a couple of related questions. If $k \subseteq L$ are fields, or more generally, division rings, with $1 < [L:k]_{\mathrm{right}} < \infty$, and $\mathcal{U}$ is any L-linear ultrafilter on a right L-vector-space X, then we may look at the k-linear filter $\mathcal{U}_k$ on X that $\mathcal{U}$ generates, and ask whether this will in general be a k-linear ultrafilter. In the case where $\mathcal{U}$ is a *principal* ultrafilter, hence generated by a subspace $xL \subseteq X$ of dimension 1, the answer is no: just as $E(\mathcal{U})$ can be identified with $\mathrm{End}_L(xL) \cong L$, so $E(\mathcal{U}_k)$ is isomorphic to $\mathrm{End}_k(xL)$, which is the matrix ring $M_{[L:k]_{\mathrm{right}}}(k)$, and not a division ring; so $\mathcal{U}_k$ cannot be a linear ultrafilter. It is also not hard to verify that if $\mathcal{U} = UL + \mathcal{C}$ where U is a Ramsey ultrafilter on a basis S of X (assumed countable-dimensional), then the k-subspace $Sk \subseteq X$ neither belongs to the k-linear filter $\mathcal{U}_k$ generated by $\mathcal{U}$, nor has a linear complement in $\mathcal{U}_k$, so in this case also, $\mathcal{U}_k$ is not a linear ultrafilter.

There is a reverse construction. If $\mathcal{F}$ is a k-linear filter on a k-vector-space X, it clearly induces an L-linear filter $\mathcal{F} \otimes_k L$ on $X \otimes_k L$. If $\mathcal{U}$ is a k-linear ultrafilter, will $\mathcal{U} \otimes_k L$ in general be an L-linear ultrafilter? If $\mathcal{U}$ is principal, it is easy to see that it will. But in the nonprincipal case, the construction of Theorem 24 (under the set-theoretic hypothesis of that Theorem) leads to examples where it is not. Indeed, let the k of that Theorem be a field, and the L a finite-dimensional division algebra over k. Then we find that for any k-linear filter $\mathcal{F}$, $E(\mathcal{F} \otimes_k L) \cong E(\mathcal{F}) \otimes_k L$. Hence if we choose $\mathcal{U}$ as in that Theorem so that $E(\mathcal{U})$ contains a copy of L^{op}, then $E(\mathcal{U} \otimes_k L)$ will contain a copy of $L^{\mathrm{op}} \otimes_k L$, which in general has zero divisors, so that $E(\mathcal{U} \otimes_k L)$ cannot be a division ring, and $\mathcal{U} \otimes_k L$ cannot be an L-linear ultrafilter.

Above, we took k to be a field and L to be a k-algebra. We remark that unless L is at least assumed to be spanned over k by k-centralizing elements, it seems curiously difficult to describe $E(\mathcal{F} \otimes_k L)$ in terms of $E(\mathcal{F})$, or, for that matter, $\mathrm{End}_L(X \otimes_k L)$ in terms of $\mathrm{End}_k(X)$.

Without set-theoretic hypotheses such as the Continuum Hypothesis or the existence of Ramsey ultrafilters, we do not know whether the above constructions $\mathcal{U} \mapsto \mathcal{U}_k$ and $\mathcal{U} \mapsto \mathcal{U} \otimes_k L$ ever take nonprincipal linear ultrafilters over one field or division ring to linear ultrafilters over the other, or ever fail to.

In introducing the concept of linear ultrafilter, we noted the analogy with set-theoretic ultrafilters. Now set-theoretic ultrafilters are of great importance because they allow the construction of *ultraproducts* and *ultrapowers*, which preserve first-order properties holding ''almost everywhere'' with respect to the ultrafilter [**9**, Cor. 11.6], [**6**, §V.5]. It is natural to ask whether linear ultrafilters have a similar property; but it is not clear what an ultraproduct or ultrapower with respect to a linear ultrafilter $\mathcal{U}$ might mean. It would seem that $E(\mathcal{U})$ should at least have the role of the ''ultrapower'' of k; and it does at least inherit from k the property of being a division ring; but we have seen that it need not inherit such basic properties as being commutative. So if there is a wide class of properties that carry over from the base rings to these eigenrings, it must be rather subtle.

We have obtained above the main results of this note. The next two sections are short tangential discussions. The final three sections look at filters and cofilters of submodules of a module X over a general ring R, and show that, after some non-obvious adjustment, one can again obtain division rings from structures corresponding to linear ultrafilters and co-ultrafilters.

5. On filters, cofilters, and iterated duals. Hales [**11**] has noted that the double dual X^{**} of a vector space X can also be regarded as a linear analog of the set $\beta(S)$ of ultrafilters on a set S. Indeed, the natural map $S \to \beta(S)$ is the universal map of the discrete space S into a compact Hausdorff space (its Stone-Čech compactification), while the map $X \to X^{**}$ is the universal map of the discrete vector space X into a vector space with a linearly compact topology.

There appear to be some connections between the objects studied in [**11**] and in the present note. Suppose that for every subspace $A \subseteq X$ we identify A^{**} with its image in X^{**} under the embedding $A^{**} \to X^{**}$ induced by the inclusion $A \to X$. Then one can show that for $\lambda \in X^{**}$, the set of subspaces $A \subseteq X$ such that $\lambda \in A^{**}$ is a linear filter $\mathcal{F}_{\lambda}$. If $0 \neq \lambda \in X \subseteq X^{**}$, then $\mathcal{F}_{\lambda}$ is the principal linear ultrafilter generated by $\lambda k \subseteq X$. When $\lambda \in X^{**} \setminus X$, one might expect $\mathcal{F}_{\lambda}$ to be a nonprincipal linear ultrafilter. But this is never so; for we can find an element $h \in X^{*}$ which is not annihilated by λ, and we see that $\ker(h)$ will not belong to $\mathcal{F}_{\lambda}$, so $\mathcal{F}_{\lambda}$ does not contain all subspaces of X of finite codimension. It would nevertheless be interesting to examine the eigenrings $E(\mathcal{F}_{\lambda})$.

If U is a set-theoretic ultrafilter on a basis S of X, and the division ring k is a finite field, Hales notes a construction of an element $\lambda(U) \in X^{**}$ from U. In this case, one finds that $\mathcal{F}_{\lambda(U)} = Uk$; hence if U is a Ramsey ultrafilter and $\lambda = \lambda(U)$, Proposition 18 above shows that $\mathcal{F}_{\lambda} \vee \mathcal{C}$ is a linear ultrafilter. We do not know whether there are any other sorts of elements $\lambda \in X^{**}$ for which this is true.

(In [**12**], the results of [**11**] are applied in the case where k is a prime field $\mathbf{Z}/p\mathbf{Z}$ to obtain invariants of infinite abelian p-groups.)

Let us also observe that if B is any subspace of X^{**}, then the subspaces $A \subseteq X$ such that $A^{**} \subseteq B$ form a linear *cofilter* $\mathcal{G}_B$ on X. This cofilter will contain all finite-dimensional subspaces of X if and only if B contains the image of X in X^{**}; so when this inclusion holds, it is plausible that $\mathcal{G}_B$ might be a nonprincipal linear co-ultrafilter. The case where this seems most likely to be so is when B is also maximal among proper subspaces of X^{**}. Maximal proper subspaces $B \subseteq X^{**}$ correspond bijectively to 1-dimensional subspaces $\mu k \subseteq X^{***}$, and in this situation B will contain the image of X if and only if μ lies in the annihilator of that image, which is a canonical linear complement of the image of X^{*} in X^{***}. So linear co-ultrafilters on X may correspond to some or all members of this linear complement. In any case, the map described above from subspaces of X^{**} to linear cofilters on X, together with the map the other way, taking each linear cofilter $\mathcal{G}$ to the subspace $\cup_{A \in \mathcal{G}} A^{**}$ (which together constitute a Galois connection), seem worthy of investigation.

The observations of the above paragraph also suggest that linear *ultrafilters* may be related to elements of X^{****} rather than, or in addition to, elements of X^{**}. We leave these ideas for others to investigate.

6. Non-linear analogs of the eigenring. Given a filter F of subsets of a set S, we may call a map $f\colon S \to S$ ''continuous with respect to F'' if inverse images under f of members of F belong to F. Such maps form a semigroup with 1 (a monoid) $R(F)$, and we may write $E(F)$ for the factor-semigroup of $R(F)$ by the congruence that identifies two such maps if they agree on a member of F. Proposition 6 suggests that for U a set-theoretic ultrafilter, $E(U)$ should be a group. It is, but not a very interesting one:

Lemma 25. *If U is an ultrafilter on a set S, then $E(U)$, defined as above, is the trivial group.*

Proof. Let $f \in R(U)$. By Lemma 12, either the fixed set of f belongs to U, or there is a $T \in U$ which is disjoint from $f(T)$. In the latter case, U would contain $f^{-1}(T) \cap T = \varnothing$, a contradiction, therefore the former condition holds, hence the image of f in $E(U)$ is the identity. □

We might hope to get more interesting groups by generalizing our context from the Boolean ring of subsets of a set S to a general Boolean ring B; here the natural generalization of an ultrafilter on S is the complement $\mathbf{m}^{\mathrm{c}}$ of a maximal ideal $\mathbf{m}$ of B, and the analogs of set-maps $f: S \to S$ are endomorphisms of B. But this context is too general: if $\mathbf{m}$ is such a maximal ideal, the semigroup $E(\mathbf{m}^{\mathrm{c}})$ need not be a group. For instance, suppose B is the Boolean ring of all finite and cofinite subsets of the natural numbers ω, and $\mathbf{m}$ the ideal of finite subsets of ω. The endomorphisms of B ''continuous with respect to $\mathbf{m}^{\mathrm{c}}$'' correspond to the maps $f: \omega \to \omega$ such that the inverse image under f of every singleton is finite, and the semigroup $E(\mathbf{m}^{\mathrm{c}})$ consists of the ''germs at ∞'' of such maps. To see that this semigroup is not a group, note that it contains the nonidentity idempotent arising from the map f that fixes every even integer, and adds 1 to every odd integer.

Whether there are intermediate degrees of generality which lead to theories that look more like what we have for linear ultrafilters, we leave for the interested reader to investigate.

7. Generalization to module categories. If X is a right module over a ring k, one may define filters, cofilters, ultrafilters and co-ultrafilters of submodules of X, and the associated rings $R(\mathcal{F})$, $R(\mathcal{G})$, etc., as above. However, the factor-rings $E(\mathcal{U}) = R(\mathcal{U})/I(\mathcal{U})$ and $E(\mathcal{V}) = R(\mathcal{V})/I(\mathcal{V})$ will generally not be division rings for $\mathcal{U}$ an ultrafilter or $\mathcal{V}$ a co-ultrafilter. The fact which we used about vector spaces that fails for modules over general rings is that every subspace has a linear complement. (Cf. third sentence of proof of Proposition 6.)

However, we shall see below that we can get division rings from such ultrafilters and co-ultrafilters using a more subtle generalization of the construction of the preceding sections. We shall develop these results below in the still more general context of modules given with *both* a filter and a cofilter. (This set-up could be generalized still further by replacing our module categories by arbitrary abelian categories.)

Definition 26. *For the remainder of this note, R will be a fixed ring.*

A filter $\mathcal{F}$ of submodules *of a right R-module X (briefly, a ''filter on X'') will mean a nonempty family of submodules of X that is closed under pairwise intersections and under passing to larger submodules, while a* cofilter $\mathcal{G}$ *of submodules will mean a nonempty family closed under pairwise sums and under passing to smaller submodules.*

If X is a right R-module and A a submodule, then for any filter $\mathcal{F}$ or cofilter $\mathcal{G}$ on X, we shall denote by $\mathcal{F}_A$, respectively $\mathcal{G}_A$, the filter or cofilter of submodules of A gotten by intersecting the submodule A with all members of $\mathcal{F}$, respectively, with all members of $\mathcal{G}$. Likewise, we will write $\mathcal{F}_{X/A}$, respectively $\mathcal{G}_{X/A}$ for the filter or cofilter of images in X/A of members of $\mathcal{F}$, respectively $\mathcal{G}$.

(That $\mathcal{F}_A$ is closed under pairwise intersections is immediate. To verify that it is closed under passing to larger submodules, suppose $B \in \mathcal{F}$, and C is any submodule of A containing $B \cap A \in \mathcal{F}_A$. Then $\mathcal{F}_A$ also contains $(B+C) \cap A = C$. Thus, $\mathcal{F}_A$ is a filter on A, as claimed. That $\mathcal{G}_A$ is a cofilter on A follows directly from the fact that it can be described as the set of those members of $\mathcal{G}$ that are submodules of A. Dual observations show that $\mathcal{F}_{X/A}$ and $\mathcal{G}_{X/A}$ are likewise filter and cofilter.)

We shall now construct a category having for objects all 3-tuples $(X, \mathcal{F}, \mathcal{G})$, where X is a right R-module, $\mathcal{F}$ a filter, and $\mathcal{G}$ a cofilter on X. Given two such objects $(X, \mathcal{F}, \mathcal{G})$ and $(Y, \mathcal{F}', \mathcal{G}')$, let us define a *premorphism* $f: (X, \mathcal{F}, \mathcal{G}) \to (Y, \mathcal{F}', \mathcal{G}')$ to mean a module homomorphism f from a submodule $X_f \in \mathcal{F}$ to a factor-module Y/Y^f with $Y^f \in \mathcal{G}'$, such that the inverse image under f of every member of $\mathcal{F}'_{Y/Y^f}$ lies in $\mathcal{F}_{X_f}$, and the image under f of every of every member of $\mathcal{G}_{X_f}$ lies in $\mathcal{G}'_{Y/Y^f}$. We shall call two premorphisms f, g: $(X, \mathcal{F}, \mathcal{G}) \to (Y, \mathcal{F}', \mathcal{G}')$ *equivalent*, written $f \sim g$, if there exist $A \in \mathcal{F}$ contained in $X_f \cap X_g$ and $B \in \mathcal{G}'$ containing $Y^f + Y^g$, such that on composing f and g on the right with the inclusion maps from A to X_f and X_g respectively, and on the left with the canonical maps from Y/Y^f and Y/Y^g to Y/B respectively, we get the same map $A \to Y/B$.

Recall that for any sets X and Y, a ''relation $r: X \to Y$'' means a subset $r \subseteq X \times Y$, and that the composite of this with a relation $s: Y \to Z$ is defined in a way generalizing the definition of composition of functions, namely,

$$s\,r \;=\; \{(x,z) \mid (\exists y \in Y)\ (x,y) \in r,\ (y,z) \in s\}.$$

It is well-known (and easily checked) that composition of relations is associative.

Now to every premorphism $f\colon (X,\ F,\ G) \to (Y,\ F',\ G')$ let us associate the relation f_{rel} defined by

$$f_{\mathrm{rel}} \;=\; \{(x,y)\in X\times Y \mid x\in X_f \text{ and } f(x) = q(y)\},$$

where $q\colon Y \to Y/Y^f$ is the canonical surjection. It is not hard to see that a subset $r \subseteq X\times Y$ is of the form f_{rel} for some premorphism f as above if and only if

(i) r is a submodule of $X\times Y$,

(ii) for every $A\in F'$, the projection of $(X\times A)\cap r$ on X is a member of F, and

(iii) for every $A\in G$, the projection of $(A\times Y)\cap r$ on Y is a member of G'.

In this situation, we see that f_{rel} uniquely determines X_f, Y^f, and f. It is also not hard to verify that given premorphisms $(X,\ F,\ G) \xrightarrow{f} (Y,\ F',\ G') \xrightarrow{g} (Z,\ F'',\ G'')$, the composite relation $g_{\mathrm{rel}}\, f_{\mathrm{rel}}$ also satisfies (i)-(iii) above, hence has the form $(gf)_{\mathrm{rel}}$ for a premorphism $gf\colon (X,\ F,\ G) \to (Z,\ F'',\ G'')$ (which one can write down explicitly). We find that this composition of premorphisms respects our equivalence relation $\sim$; thus we can make

Definition 27. *We shall denote by* $(\mathbf{Mod}_R)_{\text{filt+cofilt}}$ *the category having for objects all* 3*-tuples* $(X,\ F,\ G)$ *where* X *is a right* R*-module,* F *a filter of submodules of* X*, and* G *a cofilter of submodules of* X*, having for morphisms* $(X,\ F,\ G) \to (Y,\ F',\ G')$ *the equivalence classes* $[f]$ *of premorphisms* f *between these objects under the equivalence relation* $\sim$ *defined above, and having for composition of morphisms the operation induced by composition of the corresponding relations* f_{rel}.

The additive structure on homomorphisms among R-modules also extends to premorphisms. Given premorphisms $f,g\colon (X,\ F,\ G) \to (Y,\ F',\ G')$, we define $f+g$ to be the premorphism satisfying

$$(f+g)_{\mathrm{rel}} \;=\; \{(x,y+y') \mid (x,y)\in f_{\mathrm{rel}},\ (x,y')\in g_{\mathrm{rel}}\}.$$

For each $A\in F$, $B\in G$, this operation makes the set of premorphisms with domain A and codomain Y/B an abelian group (a subgroup of $\mathbf{Mod}_R(A,\ Y/B)$). Though the set of *all* premorphisms $(X,\ F,\ G) \to (Y,\ F',\ G')$, i.e., the union of these groups, is only a semigroup under $+$, when we divide out by the equivalence relation $\sim$ to get the desired hom-set in $(\mathbf{Mod}_R)_{\text{filt+cofilt}}$, we again get a group, which can be described as the direct limit of the above groups of premorphisms over the directed ordered set $F^{\mathrm{op}}\times G$. Note that a morphism $[f]$ is zero in this group if and only if the image under f of some member of F is a member of G'.

Let us observe that every right R-module X has a least filter of submodules, the singleton $\{X\}$, and a least cofilter of submodules, the singleton $\{\{0\}\}$. This leads to three natural subcategories of $(\mathbf{Mod}_R)_{\text{filt+cofilt}}$: We shall denote by $(\mathbf{Mod}_R)_{\text{filt}}$ the full subcategory of all objects of the form $(X,\ F,\ \{\{0\}\})$, and by $(\mathbf{Mod}_R)_{\text{cofilt}}$ the full subcategory of all objects of the form $(X,\ \{X\},\ G)$, while we note that the full subcategory of all objects of the form $(X,\ \{X\},\ \{\{0\}\})$ can be identified with

$\mathbf{Mod}_R$ itself. (When working extensively with these subcategories, it would, of course, be natural to abbreviate $(X, F, \{\{0\}\})$ to (X, F), and $(X, \{X\}, G)$ to (X, G). But as our main focus here will be on $(\mathbf{Mod}_R)_{\mathrm{filt+cofilt}}$, we shall not use those abbreviations.)

It is not hard to see that if F is a filter or G a cofilter on X, then the ring $E(F)$ or $E(G)$, defined formally exactly as in §1, can be identified with the *subring* of $\mathrm{End}(X, F, \{\{0\}\})$ or $\mathrm{End}(X, \{X\}, G)$ consisting of those morphisms that can be represented by premorphisms with X itself as domain and codomain (i.e., with $X_f = X$, respectively $X^f = \{0\}$). In the special case where R is a division ring, the fact that any subspace X_f or X^f has a linear complement in X allows one to extend any premorphism $f: (X, F, G) \to (X, F, G)$ to an endomorphism of X itself, and hence to conclude that this subring is the full endomorphism ring of our object. But this is not true for general R.

Let us record a few easy observations. Given any object (X, F, G) of $(\mathbf{Mod}_R)_{\mathrm{filt+cofilt}}$ and any $A \in F$, the inclusion of A in X is easily seen to be a premorphism, which yields an *isomorphism*

$$(A, F_A, G_A) \xrightarrow{\sim} (X, F, G). \tag{7}$$

In the same way, for any $B \in G$, we get a natural isomorphism

$$(X, F, G) \xrightarrow{\sim} (X/B, F_{X/B}, G_{X/B}). \tag{8}$$

A recurring technique below will be to compose a given morphism on the right or the left with such isomorphisms or their inverses, so as to bring it to a more convenient form.

Note that in the situation of (7), if the given filter F is principal, with generator A, then we have $F_A = \{A\}$, so that (X, F, G) is isomorphic to an object of $(\mathbf{Mod}_R)_{\mathrm{cofilt}}$, namely $(A, \{A\}, G_A)$. Similarly, if G is principal with generator B, then (8) shows that (X, F, G) is isomorphic to $(X/B, F_{X/B}, \{\{0\}\})$, an object of $(\mathbf{Mod}_R)_{\mathrm{filt}}$.

The object $(\{0\}, \{\{0\}\}, \{\{0\}\})$ is easily seen to be a ''zero object'' (initial-final object) of $(\mathbf{Mod}_R)_{\mathrm{filt+cofilt}}$. Calling this object ''0'', we find that a general object (X, F, G) is isomorphic to 0 if and only if $F \cap G \neq \varnothing$. (If that intersection is nonempty, then the identity endomorphism of (X, F, G) is equal to the zero map, since it takes a member of F to a member of G; hence the unique maps $(X, F, G) \rightleftarrows 0$ are inverses. The converse is seen similarly.) Thus we shall call (X, F, G) a ''nonzero object'' if F and G are disjoint.

Given two filters F and F' and two cofilters G and G' on a module X, the identity map $1_X: X \to X$ is a premorphism $(X, F, G) \to (X, F', G')$ if and only if $F \supseteq F'$ and $G \subseteq G'$. If $F = F'$ and $G = G'$, then the resulting morphism $[1_X]$ is an identity morphism of our category, hence invertible. In the general case, $[1_X]$ may or may not be invertible. (It can be invertible without its inverse arising from 1_X. For instance, take any nonzero X, and let $*$ denote the set of *all* submodules of X; this is both a filter and a cofilter. Then it is easy to see that 1_X is a premorphism $(X, *, \{X\}) \to (X, \{\{0\}\}, *)$, but that the only premorphism going the other way is the zero map $X \to X$. But since $(X, *, \{X\})$

and $(X,\ \{\{0\}\},\ *)$ are both zero objects, the morphism induced by that premorphism is indeed inverse to $[1_X]$.)

It is clear from Zorn's Lemma that given any nonzero object (X, F, G), we can find a filter $F' \supseteq F$ and a cofilter $G' \supseteq G$ *maximal* for the property of being disjoint, i.e., for the property that (X, F', G') is nonzero. Likewise, given any nonzero object $(X, F, \{\{0\}\})$ of $(\mathbf{Mod}_R)_{\mathrm{filt}}$ we can find a maximal proper filter $F' \supseteq F$, and given any nonzero object $(X, \{X\}, G)$ of $(\mathbf{Mod}_R)_{\mathrm{cofilt}}$ we can find a maximal proper cofilter $G' \supseteq G$. We shall now show that in each of these cases, the resulting object (X, F', G'), respectively $(X, F', \{\{0\}\})$, respectively $(X, \{X\}, G')$ has for its endomorphism ring a division ring. Note that the latter two cases are not instances of the first case. For instance, if F is a maximal proper filter on X, i.e., a maximal filter disjoint from the cofilter $\{\{0\}\}$, then the latter cofilter need not be maximal among cofilters disjoint from F. Indeed, if U is a nonprincipal linear ultrafilter on a vector space, then the set of finite-dimensional subspaces is a cofilter disjoint from U, and strictly larger than $\{\{0\}\}$.

We will get the above assertions on division rings of endomorphisms as instances of more general conditions for the invertibility of a morphism

$$[f] : (X, F, G) \ \to\ (Y, F', G') \tag{9}$$

in $(\mathbf{Mod}_R)_{\mathrm{filt+cofilt}}$. Let us begin with the hardest case.

Lemma 28. *If, in* (9), G *is maximal among cofilters disjoint from* F, *and* F' *is maximal among filters disjoint from* G', *then* $[f]$ *is either zero or invertible.*

Proof. It is not hard to verify that the condition that the third component of (X, F, G) is maximal among cofilters disjoint from the second component is inherited by all of the (isomorphic) objects (A, F_A, G_A) for $A \in F$, and $(X/B, F_{X/B}, G_{X/B})$ for $B \in G$; and the analogous statement is true for the maximality condition on the second component of (Y, F', G'). Thus, precomposing (9) with the isomorphism (7) with $A = X_f$, and postcomposing it with the isomorphism (8) with $B = Y^f$, we are reduced to the case where $X_f = X$ and $Y^f = \{0\}$, i.e., where the premorphism f is a module homomorphism $X \to Y$.

Suppose now that $\ker(f) \notin G$. Then by our maximality assumption on G, the cofilter $G \vee \{\ker(f)\}$ is not disjoint from F; i.e., there is some $B \in G$ such that $\ker(f) + B \in F$. Now f applied to this sum gives $f(B)$, which belongs to G'. Thus f carries a member of F to a member of G', so $[f] = 0$.

So we may assume $\ker(f) \in G$. Thus, replacing X by $X/\ker(f)$ (i.e., factoring out of (9) an isomorphism (8) with $B = \ker(f)$) we may assume f one-to-one; hence, making an identification, we may assume that $X \subseteq Y$ and that f is the inclusion map. Note that the fact that this inclusion is a premorphism says that

$$F \ \supseteq\ F'_X \qquad \text{and} \qquad G \ \subseteq\ G'. \tag{10}$$

Now suppose that $X \notin F'$. Then by our maximality assumption on F', the filter $F' \vee \{X\}$ is not disjoint from G'; i.e., there is some $A \in F'$ such that $X \cap A \in G'$. But $X \cap A \in F'_X \subseteq F$, so again our premorphism f carries a member of F to a member of G', giving $[f] = 0$.

So we are now reduced to the case $X \in F'$, and replacing Y by this submodule (i.e., factoring out an isomorphism (7)), we may assume $X = Y$.

Now if F and G' are not disjoint, we once again have $[f] = 0$, so assume they are. Then (10) and our maximality hypotheses imply that $F = F'$ and $G = G'$. So we have reduced ourselves to the case where f is an identity morphism in $(\mathbf{Mod}_R)_{\text{filt+cofilt}}$, hence invertible. □

Lemma 29. *If, in* (9), $F = \{X\}$, *while* G *and* G' *are maximal among cofilters disjoint from* F *and* F' *respectively, then* $[f]$ *is either zero or invertible.*

Proof. The proof of the preceding Lemma remains valid through (10), since maximality of F' was not used through that point. Thus we may again assume f is the inclusion of a submodule X in the module Y.

However, *before* going through that argument, let us note that if $f(X_f) \in G'$, then $[f] = 0$ and we are done. Thus we may assume the contrary; so maximality of G' (the new hypothesis in the present situation) implies that for some $B \in G'$ we have $f(X_f) + B \in F'$. Hence, let us begin by replacing Y by Y/B for this B, leading to the condition $f(X_f) \in F'$, and *then* go through the above reduction. Thus, when we are finished with this reduction, we have not only that $X \subseteq Y$, but also that $X \in F'$. Thus, we may replace Y by its submodule X, i.e., we may assume $X = Y$.

Now since F is the *least* cofilter on X, the first inclusion of (10) forces $F' = F$. The second inclusion and the maximality of G then give $G' = G$. So we have again reduced to the case where $[f]$ is an identity morphism. □

The dual argument yields

Lemma 30. *If, in* (9), $G' = \{\{0\}\}$, *and* F *and* F' *are maximal among filters disjoint from* G *and* G' *respectively, then* $[f]$ *is either zero or invertible.* □

We can now apply the above three results to endomorphism rings. (Curiously, though the statements (i)-(iii) in the next Theorem are not cases of one another, statement (iv) is a case of all of (i)-(iii).)

Theorem 31. *The endomorphism ring of an object* (X, F, G) *of* $(\mathbf{Mod}_R)_{\text{filt+cofilt}}$ *is a division ring in each of the following cases:*

(i) *If* F *is maximal among filters disjoint from* G, *and* G *is maximal among filters disjoint from* F.

(ii) *If* $G = \{\{0\}\}$, *and* F *is maximal among proper filters on* X.

(iii) *If* $F = \{X\}$, *and* G *is maximal among proper cofilters on* X.

(iv) *If* $F = \{X\}$, $G = \{\{0\}\}$, *and* X *is a simple object of* $\mathbf{Mod}_R$. □

8. Some examples. The construction of nonprincipal linear ultrafilters and co-ultrafilters on a vector space, studied in the first part of this note, required Zorn's Lemma. But there are interesting examples in our more general module-theoretic framework that are quite explicit.

Example 32. *Fields and division rings of fractions.* Suppose R is a commutative integral domain, or more generally, a right Ore domain [**8**, §1.3]. Let us consider R as a right module over itself, and let $\mathcal{F}$ be the set of *all* its nonzero right ideals. The Ore condition insures that $\mathcal{F}$ is closed under intersections, hence is a filter, and by definition it is maximal among proper filters of submodules; hence $\mathrm{End}((X, \mathcal{F}, \{\{0\}\}))$ is a division ring. What does this ring look like?

Given any element

$$[f]:\ (R, \mathcal{F}, \{\{0\}\}) \ \to\ (R, \mathcal{F}, \{\{0\}\})$$

of this ring, the representing premorphism f will be a module homomorphism from a nonzero right ideal of R into R. Restricting if necessary to a subideal of its domain R_f, we can assume without loss of generality that R_f is a principal right ideal sR $(s\in R\setminus\{0\})$. Let f carry the generator s of this ideal to $a\in R$; then it carries each element sx to ax; it can thus be regarded as ''formal left multiplication by as^{-1}''.

Conversely, for any $a\in R$ and $s\in R\setminus\{0\}$, we claim that the module homomorphism $sR\to R$ taking s to a is a pre-endomorphism of $(R, \mathcal{F}, \{\{0\}\})$. The one non-obvious verification is that this map is continuous with respect to $\mathcal{F}$. Given any nonzero right ideal of R, without loss of generality a principal right ideal tR, we need to know that its inverse image under our homomorphism is nonzero; i.e., that the equation $tx = ay$ has nonzero solutions (x,y); but this follows from the right Ore assumption on R. If, as above, we denote this premorphism by as^{-1}, then it is not hard to verify that our equivalence relation among premorphisms corresponds to the equivalence relation on symbols as^{-1} used in defining the division ring of right fractions of R, and that the ring structure of $\mathrm{End}((R, \mathcal{F}, \{\{0\}\}))$ is precisely that of this division ring.

Note that if, in the above situation, we define $R(\mathcal{F})$ and $I(\mathcal{F})$, as in §1, to consist of endomorphisms of the *whole* right module R that are continuous, respectively trivial, with respect to $\mathcal{F}$, then $R(\mathcal{F})\cong R$ and $I(\mathcal{F}) = \{0\}$, so that $E(\mathcal{F}) = R(\mathcal{F})/I(\mathcal{F}) \cong R$; that is, $E(\mathcal{F})$ is the canonical copy of R within its division ring of fractions.

Example 33. *Localization with respect to Ore denominator sets.* Suppose, more generally, that we take a ring R and a set S of elements of R, and let $\mathcal{F}$ denote the set of those right ideals of R that contain at least one member of S. When will this be a filter, and when can the endomorphism ring of the corresponding object of $(\mathbf{Mod}_R)_{\mathrm{filt}}$ be regarded as a ''localization of R''?

The condition for $\mathcal{F}$ to be a filter is clearly that every two elements of S have a common right multiple in S. If we are to consider the endomorphism ring of our object to be a localization of R, then we would like to map R into it by sending each $a\in R$ to the right R-module endomorphism of R given by left multiplication by a; so we want this to be continuous with respect to $\mathcal{F}$; i.e., the inverse image of each sR $(s\in S)$ under left multiplication by a should contain some $s'R$

$(s' \in S)$. In other words, for every $s \in S$ and $a \in R$ there should exist $s' \in S$ and $a' \in R$ such that $sa' = as'$. This is called the ‘‘right Ore condition’’ on $S \subseteq R$ [**8**, p.15, (D.2)].

Let us assume now that the elements of S are all left non-zero-divisors. Then for each $s \in S$ we have a module homomorphism $sR \to R$ taking s to 1, and we would like these to be premorphisms with respect to $\mathcal{F}$, representing elements ‘‘s^{-1}’’. It is easy to check that these maps are premorphisms if and only if for all $s, s' \in S$, the set S contains a right multiple of ss'. But as far as the effect on our filter $\mathcal{F}$ is concerned, this is equivalent to requiring S to contain ss' itself, i.e., to requiring S to be a multiplicative semigroup. This is condition (D.1) of [**8**, p.15], and together with condition (D.2) assumed above, and our assumption that S consists of left non-zero-divisors (which vacuously gives the condition (D.3), *op. cit.*), this makes S a *right denominator set* [**8**, p.16]. Note, moreover, that if for $s_0, s_1 \in S$ we apply (D.2) with these elements as a and s, then (D.1) tells us that the common value of the two sides of the equation given by (D.2) lies in S. This yields the one other condition we assumed above, namely that every two members of S have a common right multiple also in S. Thus, we do not have to assume that separately.

For S satisfying these conditions, $\mathrm{End}((R, \mathcal{F}, \mathbf{\{}\{0\}\mathbf{\}}))$ is precisely the classical ring of right fractions RS^{-1}.

(What if we don’t exclude left zero-divisors from S? If $s \in S$ is a left zero-divisor, we do not in general have a right module homomorphism $sR \to R$ taking s to 1; instead, we get a homomorphism $sR \to R/\mathrm{Ann}_{\mathrm{right}}(s)$, where $\mathrm{Ann}_{\mathrm{right}}(s)$ denotes the right annihilator of s. This suggests that we should define a cofilter $\mathcal{G}$ containing these right annihilator ideals, and look at the endomorphism ring of $(R, \mathcal{F}, \mathcal{G})$. However, to make the operations of left multiplication by elements $a \in R$ premorphisms, we must also require $\mathcal{G}$ to contain all product ideals $a\,\mathrm{Ann}_{\mathrm{right}}(s)$; and to obtain the other required conditions, still more assumptions must be added, making this approach awkward. Note, however, that the premorphism of left multiplication by a is equivalent to zero if it annihilates a member of $\mathcal{F}$, i.e., if a lies in $\mathrm{Ann}_{\mathrm{left}}(s)$ for some $s \in S$. The right Ore condition and the multiplicative closure assumption guarantee that the these *left* annihilator ideals form a directed system, whose union is a 2-sided ideal I. The general Ore localization construction given in [**8**] uses one further assumption, (D.3), namely that every *right* annihilator ideal $\mathrm{Ann}_{\mathrm{right}}(s)$ is contained in that ideal I; the construction is then equivalent to that of $\mathrm{End}((R, \mathcal{F}, \mathcal{G}))$ where $\mathcal{G}$ is the principal cofilter generated by I. This is equivalent to the construction of the preceding paragraph, performed over the ring R/I, in which the image of S consists of non-zero-divisors.)

Another classical construction over an arbitrary ring R which is equivalent to that of the endomorphism ring of an appropriate object $(R, \mathcal{F}, \mathbf{\{}\{0\}\mathbf{\}})$ is that of the *maximal ring of quotients* of R; here $\mathcal{F}$ is the filter of all *dense* right ideals of R. This characterization of that construction can be deduced from statements ii-iv of [**17**, Theorem 24.8].

What about endomorphism rings of objects $(X, \{X\}, \mathcal{G})$ determined by cofilters $\mathcal{G}$?

There are no nontrivial examples with X Noetherian, since in a Noetherian module, every cofilter of right ideals will have a maximal element, and so be principal. Let us therefore look at one of the simplest examples of a non-Noetherian module over a familiar ring.

Example 34. *Some p-adics.* Let $R = \mathbf{Z}$, let p be a prime, and consider the $\mathbf{Z}$-module (abelian group) Z_{p^∞}, i.e., the p-torsion part of $\mathbf{Q}/\mathbf{Z}$. The proper submodules of Z_{p^∞} are the finite abelian groups Z_{p^n}, which form an ascending chain; this chain constitutes the unique nonprincipal cofilter of submodules of Z_{p^∞}, and is clearly a maximal proper cofilter; let us call it $\mathcal{G}$.

Every endomorphism of Z_{p^∞} carries $\mathcal{G}$ into itself, and no nonzero endomorphism is trivial with respect to $\mathcal{G}$; hence the ring $E(\mathcal{G})$ is just the endomorphism ring of Z_{p^∞}, which is the ring of *p-adic integers*. A general element of $\mathrm{End}((Z_{p^\infty}, \{Z_{p^\infty}\}, \mathcal{G}))$, represented by a premorphism $Z_{p^\infty} \to Z_{p^\infty}/Z_{p^n}$, is essentially p^{-n} times an endomorphism of Z_{p^∞}, so $\mathrm{End}((Z_{p^\infty}, \{Z_{p^\infty}\}, \mathcal{G}))$ can be identified with the field of *p-adic rationals*.

9. Three remarks, and an unused lemma.

Remark 35. *A couple of concepts from the literature.* In the study of *torsion theories* on R-modules [**10**], one considers filters of *two-sided ideals* of R. Such filters are equivalent to filters $\mathcal{F}$ of right ideals that are generated by the two-sided ideals they contain; hence they yield objects $(R, \mathcal{F}, \{\{0\}\}) \in (\mathbf{Mod}_R)_{\mathrm{filt}}$. Moreover, an observation basic to their use in relation to torsion theories is that for any right R-module X, one gets a filter $X\mathcal{F}$ of submodules of X. This suggests looking at the objects $(X, X\mathcal{F}, \{\{0\}\}) \in (\mathbf{Mod}_R)_{\mathrm{filt}}$. The authors leave to the interested reader the question of the relation between this construction, and the standard construction of localizing the category $\mathbf{Mod}_R$ with respect to the given filter.

In *model theory* there is the concept of an *Abelian Structure*, meaning an abelian group A given with a set of subgroups of A^n for each integer n, called ''definable'' subgroups, satisfying appropriate closure conditions [**14**]. In certain cases, the definable subgroups of the original abelian group $A^1 = A$ form a filter (although the Abelian Structure contains much more information than just the definable subgroups of A^1). Again, the authors have not investigated possible relations between the theory of Abelian Structures and the ideas of this note, which the interested reader is welcome to pursue.

Remark 36. *A puzzle.* We mentioned in §1 that for X a vector space, and $\mathcal{F}$ or $\mathcal{G}$ a linear filter or cofilter on X, the ring $E(\mathcal{F}) = \mathrm{End}((X, \mathcal{F}, \{\{0\}\}))$, respectively $E(\mathcal{G}) = \mathrm{End}((X, \{X\}, \mathcal{G}))$ could be interpreted as the endomorphism ring of a certain left or right $\mathrm{End}(X)$-module. For a general ring R, we would not expect there to be such easy characterizations of endomorphism rings of objects of $(\mathbf{Mod}_R)_{\mathrm{filt}}$ or $(\mathbf{Mod}_R)_{\mathrm{cofilt}}$. But in the case of vector spaces over a field or division ring, since there exist such interpretations for endomorphism rings of objects of $(\mathbf{Mod}_k)_{\mathrm{filt}}$ and $(\mathbf{Mod}_k)_{\mathrm{cofilt}}$, we hoped to find such an interpretation for endomorphism rings of objects of $(\mathbf{Mod}_R)_{\mathrm{filt+cofilt}}$ as well. We were not able to, and are still curious as to whether one exists.

Remark 37. *More on maximal filter-cofilter pairs on a vector space.* Suppose $\mathcal{U}$ is a linear ultrafilter on a vector space X over a division ring k, i.e., a maximal member of the set of linear filters disjoint from the trivial cofilter $\{\{0\}\}$. We observed earlier that $\{\{0\}\}$ will not in general be maximal among linear cofilters disjoint from $\mathcal{U}$. How big will a maximal cofilter $\mathcal{G}$ disjoint from $\mathcal{U}$ be?

If $\mathcal{U}$ is a principal linear ultrafilter, generated by a 1-dimensional subspace xk, then it is straightforward to see that $\mathcal{G}$ must be the principal linear co-ultrafilter generated by a linear complement to this space. If $\mathcal{U}$ is nonprincipal, can we still find cases where $\mathcal{G}$ is a linear co-ultrafilter? In other words, can a nonprincipal linear ultrafilter and a nonprincipal linear co-ultrafilter be disjoint? Assuming the existence of a q-point (§§2-3 above), the answer is yes: If U is a q-point, and $\mathcal{U}$ any linear ultrafilter containing the linear filter Uk, then it is easy to show that $\mathcal{U}$ and the linear co-ultrafilter $U^{\mathrm{c}}k$ are disjoint.

Can one get examples without such set-theoretic assumptions? In the opposite direction, can a linear cofilter maximal for the property of being disjoint from a given nonprincipal linear ultrafilter *fail* to be a linear co-ultrafilter? Can it be very small, e.g., consist only of the subspaces of finite codimension? Are there maximal disjoint pairs of a linear filter $\mathcal{F}$ and linear cofilter $\mathcal{G}$ where neither of these is a linear (co)ultrafilter? The authors have not investigated these questions.

An unused lemma. The first author originally expected the categories $(\mathbf{Mod}_R)_{\mathrm{filt+cofilt}}$, $(\mathbf{Mod}_R)_{\mathrm{filt}}$, and $(\mathbf{Mod}_R)_{\mathrm{cofilt}}$ to be abelian categories, and that Theorem 31 (i)-(iii) would be instances of the general fact that the endomorphism ring of a *simple* object of an abelian category is a division ring. However, though these categories are **Ab**-categories (i.e., have abelian group structures on their hom-sets such that composition of morphisms is bilinear), they turn out not to be abelian categories, even when R is a field.

The next hope was that the objects treated in that Theorem would satisfy some conditions in their home categories which in an abelian category were equivalent to simplicity, and that their home categories would have enough of the properties of abelian categories to imply that the endomorphism rings of objects with these properties were division rings. The following conditions on a nonzero object M of an **Ab**-category seemed good candidates for such ''simplicity'' conditions.

(**S1**) Every nonzero morphism with codomain M is an epimorphism.

(**S2**) Every monomorphism from a nonzero object to M is an isomorphism.

(**S3**) Every nonzero morphism with domain M is a monomorphism.

(**S4**) Every epimorphism from M to a nonzero object is an isomorphism.

It is an easy exercise to verify

Lemma 38. *Let* **C** *be an* **Ab**-*category. Then*

(i) *If in* **C** *every morphism factors as* mono ∘ epi, *then as conditions on an object of* **C**, (**S 2**)⇒(**S 1**) *and* (**S 4**)⇒(**S 3**).

(ii) *If in* **C** *every morphism that is both epic and monic is invertible, then as conditions on an object of* **C**, (**S 1**)⇒(**S 2**) *and* (**S 3**)⇒(**S 4**).

(iii) *If in* **C** *every morphism has a kernel, then as conditions on an object of* **C**, (**S 1**)⇒(**S 3**) *and* (**S 2**)⇒(**S 3**).

(iv) *If in* **C** *every morphism has a cokernel, then as conditions on an object of* **C**, (**S 3**)⇒(**S 1**) *and* (**S 4**)⇒(**S 1**).

(v) *Every nonzero morphism in* **C** *from an object satisfying* (**S 3**) *to an object satisfying* (**S 2**) *is an isomorphism. In particular, the endomorphism ring of a nonzero object satisfying both* (**S 3**) *and* (**S 2**) *is a division ring.*

(vi) *Every nonzero morphism in* **C** *from an object satisfying* (**S 4**) *to an object satisfying* (**S 1**) *is an isomorphism. In particular, the endomorphism ring of a nonzero object satisfying both* (**S 4**) *and* (**S 1**) *is a division ring.* □

However, it turned out that general morphisms in the categories $(\mathbf{Mod}_R)_{\text{filt+cofilt}}$ do not have kernels or cokernels, and can be epic and monic without being invertible, so that (ii), (iii) and (iv) are irrelevant to our situation. One can show that morphisms do have factorizations as mono ∘ epi, so by (i) we have the implications (**S 2**)⇒(**S 1**) and (**S 4**)⇒(**S 3**); but objects (X, F, G) in which F is maximal among filters disjoint from G and G is maximal among cofilters disjoint from F in general satisfy only the conditions (**S 1**) and (**S 3**), which do not alone imply that the endomorphism ring of an object is a division ring. Thus, Theorem 31 had to be proved without recourse to the above approach.

Some results which the first author obtained while attempting this approach are recorded in [**4**]; in particular, characterizations of epimorphisms and monomorphisms in the categories $(\mathbf{Mod}_R)_{\text{filt+cofilt}}$, $(\mathbf{Mod}_R)_{\text{filt}}$ and $(\mathbf{Mod}_R)_{\text{cofilt}}$, and arguments and examples establishing the positive and negative assertions of the preceding paragraph.

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