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CONTINUITY OF HOMOMORPHISMS ON PRO-NILPOTENT ALGEBRAS

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ABSTRACT. Let $\mathbf{V}$ be a variety of not necessarily associative algebras, and A an inverse limit of nilpotent algebras $A_i \in \mathbf{V}$, such that some finitely generated subalgebra $S \subseteq A$ is dense in A under the inverse limit of the discrete topologies on the A_i.

A sufficient condition on $\mathbf{V}$ is obtained for all algebra homomorphisms from A to finite-dimensional algebras B to be continuous; in other words, for the kernels of all such homomorphisms to be open ideals. This condition is satisfied, in particular, if $\mathbf{V}$ is the variety of associative, Lie, or Jordan algebras.

Examples are given showing the need for our hypotheses, and some open questions are noted.

1. Background: From pro-p groups to pro-nilpotent algebras

A result of Serre's on topological groups says that if G is a pro-p group (an inverse limit of finite p-groups) which is topologically finitely generated (i.e., has a finitely generated subgroup which is dense in G under the inverse limit topology), then any homomorphism from G to a finite group H is continuous ([6, Theorem 1.17], [14, Section I.4.2, Exercises 5–6, p. 32]). Two key steps in the proof are that (i) every finite homomorphic image of a pro-p group G is a p-group, and (ii) for G a topologically finitely generated pro-p group, its subgroup $G^p[G,G]$ is closed.

In [2], we obtained a result similar to (i), namely, that if A is a pro-nilpotent (not necessarily associative) algebra over a field k, then every finite-dimensional homomorphic image of A is nilpotent. The analog of (ii) would

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say that when such an A is topologically finitely generated, the ideal A^2 is closed. (To see the analogy, note that $G/(G^p[G,G])$ is the universal homomorphic image of G which is a $\mathbb{Z}/p\mathbb{Z}$-vector space, and A/A^2 the universal homomorphic image of A which is a k-vector space with zero multiplication.)

Is this analog of (ii) true?

The proof of the group-theoretic statement (ii) is based on first showing that if a finite p-group F is generated by $g_1, \ldots, g_r$, then

$$[F,F] = [F,g_1] \cdots [F,g_r], \tag{1}$$

i.e., every member of $[F,F]$ is a product of exactly r commutators of the indicated sorts. From this one deduces that if a pro-p group G is generated topologically by $g_1, \ldots, g_r$, then likewise $G^p[G,G] = G^p[G,g_1] \cdots [G,g_r]$. The latter set is compact since G is, hence it must be closed in G, as claimed.

If an *associative* algebra S is generated by elements $g_1, \ldots, g_r$, it is clear that, similarly,

$$S^2 = Sg_1 + \cdots + Sg_r. \tag{2}$$

We shall see below that whether a formula like (2) holds for all finitely generated S in a variety $\mathbf{V}$ of not necessarily associative algebras depends on $\mathbf{V}$. In particular, we shall find that (2) itself also holds for Lie algebras, while a more complicated relation which we can use in the same way holds for Jordan algebras. For varieties in which such identities hold, we obtain an analog of Serre's result, Theorem 11 below. Examples in Section 6 show the need for some such condition on $\mathbf{V}$, and for the topological finite generation of A.

2. Review of linearly compact vector spaces

The analog of the compact topology on the underlying set of a profinite group is the linearly compact topology (defined below) on the underlying vector space of a pro-finite-dimensional algebra. Most of the basic theory of linear compactness of topological vector spaces goes over to modules over a general associative ring, and we shall sketch the material here in that context. However, we shall only call on the vector space case, so the reader who so prefers may read this section with only that case in mind.

Definition 1. Let M be a left module over an associative unital ring R.

A *linear topology* on M means a topology under which the module operations are continuous, and which has a neighborhood basis of 0 consisting of submodules. (A basis of open sets is then given by the cosets of these open submodules.)

Under such a topology, M is said to be *linearly compact* if it is Hausdorff, and if every family of cosets of closed submodules of M that has the finite intersection property has nonempty intersection.

(We assume no topology given on R. Indeed, even if R is the real or complex field, its standard topology is unrelated to the linear topologies on R-vector-spaces.)

The closed submodules, used in the above definition of linear compactness, are characterized in the following lemma.

LEMMA 2. *Under any linear topology on a module M, the closed submodules are the (not necessarily finite) intersections of open submodules.*

Proof. Every open submodule N is closed, since its complement is the union of its nonidentity cosets, which are open. Hence, intersections of open submodules are also closed.

Conversely, if N is a closed submodule and x a point not in N, then for some open submodule M', the coset $x + M'$ is disjoint from N; equivalently, $x \notin N + M'$. Since the submodule $N + M'$ is a union of cosets of M', it is open, so N is the intersection of the open submodules containing it. □

We shall see below that the closed submodules of a linearly topologized module do not in general determine the open submodules, and hence the topology; but that they do when R is a field and M is linearly compact.

Here are some tools for proving a linear topology linearly compact.

LEMMA 3 (cf. [7, Chapter II (27.2–4), (27.6)]). *Let R be an associative unital ring.*

(i) *If an R-module M with a Hausdorff linear topology has descending chain condition on closed submodules, it is linearly compact. In particular, if an R-module M is artinian, it is linearly compact under the discrete topology.*

(ii) *A closed submodule of a linearly compact R-module is linearly compact under the induced topology.*

(iii) *The image of a linearly compact module under a continuous homomorphism into a Hausdorff linearly topologized module is again linearly compact. In particular, any Hausdorff linear topology on a module weaker than a linearly compact topology is linearly compact.*

(iv) *The limit (in the category-theoretic sense, which includes inverse limits, direct products, fixed-point modules of group actions, etc.) of any small system of linearly compact R-modules and continuous maps among them is again linearly compact. ("Small" means indexed by a set rather than a proper class.)*

Sketch of proof. The verifications of (i)–(iii) are routine. (The Hausdorffness condition in (iii) is needed only because "Hausdorff" is part of the definition of linearly compact.)

To get (iv), recall [8, proof of Theorem V.2.1, p. 109] [1, proof of Proposition 7.6.6] that if a category has products and equalizers, then it has small limits, and the limit of a small system of objects A_i and morphisms among them can be constructed as the equalizer of a pair of maps between products of copies of the A_i. Now the product topology on a direct product of linearly topologized modules is again linear, and the proof of Tychonoff's theorem adapts to show that a direct product of linearly compact modules is again linearly compact (cf. [7, Chapter II (27.2)]). The equalizer of two morphisms in the category of linearly topologized modules is the kernel of their difference, a submodule which is closed if the codomain of the maps is Hausdorff. In this situation, (ii) shows that if the domain module is linearly compact, the equalizer is also linearly compact, completing the proof of (iv). □

Note that by the second sentence of (i) above, every finite-dimensional vector space over a field k is linearly compact under the discrete topology. In particular, linear compactness does not imply ordinary compactness. Bringing in (iv), we see that an inverse limit of discrete finite dimensional vector spaces is always linearly compact.

Over a general ring R, are the artinian modules the only modules linearly compact in the discrete topology? Not necessarily: if R is a complete discrete valuation ring which is not a field, we see that as a discrete R-module, R is linearly compact. (More on this example later.)

Here are some restrictions on linearly compact topologies.

Lemma 4 (cf. [7, Chapter II (25.6), (27.5), (27.7)]). *Let R be an associative unital ring.*

(i) *If an R-module M is artinian, then the only Hausdorff linear topology on M is the discrete topology.*

(ii) *If an R-module M is linearly compact under the discrete topology, then M does not contain a direct sum of infinitely many nonzero submodules.*

(iii) *Any linearly compact submodule N of a Hausdorff linearly topologized R-module M is closed.*

(iv) *Every linearly compact R-module M is the inverse limit of an inversely directed system of discrete linearly compact R-modules.*

(v) *In a linearly compact R-module, the sum of any two closed submodules is closed.*

Sketch of proof. (i) Hausdorffness implies that $\{0\}$ is the intersection of all the open submodules of M. By the Artinian assumption, some finite intersection of these is therefore zero. A finite intersection of open submodules is open, so $\{0\}$ is open, i.e., M is discrete.

(ii) If M contains an infinite direct sum $\bigoplus_{i\in I} N_i$, and each N_i has a nonzero element x_i, then the system of cosets $C_i = x_i + \bigoplus_{j\in I-\{i\}} N_j$ $(i \in I)$ has the finite intersection property (namely, $C_{i_1} \cap \cdots \cap C_{i_n}$ contains $x_{i_1} + \cdots + x_{i_n}$), and all these sets are closed since M is discrete. But they have no element in common: such an element would lie in $\bigoplus_{i\in I} N_i$ since each C_i does, but would have to have a nonzero summand in each N_i.

(iii) Suppose $x \in M$ is in the closure of the linearly compact submodule N. Then for every open submodule $M' \subseteq M$, the coset $x + M'$ has nonempty intersection with N, and we see that these intersections form a family of cosets within N of the submodules $N \cap M'$, which are clearly closed in N. Linear compactness of N implies that these sets have nonempty intersection; but the intersection of the larger sets $x + M'$ is $\{x\}$ because M is Hausdorff. Hence $x \in N$, showing that N is closed.

(iv) Note that the open submodules $N \subseteq M$ form an inversely directed system under inclusion; let $M' = \varprojlim_N M/N$, the inverse limit of the system of discrete factor-modules, with the inverse-limit topology. The universal property of the inverse limit gives us a continuous homomorphism $f : M \to M'$.

Now each point of M' arises from a system of elements of the factor-modules M/N, equivalently, from a system of cosets of the open submodules N, having a compatibility relation that implies the finite intersection property. Using the linear compactness of M, we deduce that f is surjective. Since M is Hausdorff, the maps $M \to M/N$, as N ranges over the open sets, separate points, hence so does the single map $f : M \to M'$; so f is also injective.

Finally, every open submodule N of M is the inverse image of the open submodule $\{0\} \subseteq M/N$, hence is the inverse image of an open submodule of M'; so the topology of M is no finer than that of M', so f is an isomorphism of topological modules.

(v) Let M_1 and M_2 be closed submodules of the linearly compact module M. By (iv) of the preceding lemma, the direct product $M_1 \times M_2$ is linearly compact under the product topology. The map $M_1 \times M_2 \to M$ given by addition is continuous, hence its image, $M_1 + M_2$, is linearly compact by point (iii) of that lemma, hence closed by (iii) of this one. □

Returning to the curious case of a complete discrete valuation ring R, which we saw was linearly compact under the discrete topology, one may ask, “What about its valuation topology, which is not discrete?” The next lemma (not needed for the main results of this paper) shows that R is linearly compact under that topology as well.

LEMMA 5. *Let R be an associative unital ring, M a left R-module, and $T' \subseteq T$ Hausdorff linear topologies on M.*

Then if M is linearly compact under T, it is also linearly compact under T'. In this situation, the same submodules (but not, in general, the same sets) are closed in the two topologies.

Proof. The first assertion is the content of the second sentence of Lemma 3(iii).

Now if a submodule $N \subseteq M$ is closed under T, then by Lemma 3(ii) it is linearly compact under the topology induced on it by T, hence, by the above observation with N in place of M, also under the weaker topology induced by T'. Hence by Lemma 4(iii), it is closed in M under T'. The reverse implication follows from the assumed inclusion of the topologies, giving the second assertion of the lemma.

For R a complete discrete valuation ring which is not a field, and $M = R$, observe that $M - \{0\}$ is closed in the discrete topology but not in the valuation topology, yielding the parenthetical qualification. ☐

We now note some stronger statements that are true when R is a field. (The proofs of (i), (ii) and (iv) below work, with appropriately adjusted language, for arbitrary R if "vector space" is changed to "semisimple module", i.e., direct sum of simple modules.)

Lemma 6 (cf. [7, Chapter II (27.7), (32.1)]). *Let k be a field and V a topological k-vector-space. Then*

(i) *If V is discrete, it is linearly compact if and only if it is finite-dimensional.*

(ii) *If V is linearly compact, then its open subspaces are precisely its closed subspaces of finite codimension.*

(iii) *The following conditions are equivalent*: (a) *V is linearly compact.* (b) *V is the inverse limit of an inversely directed system of finite-dimensional discrete vector spaces.* (c) *Up to isomorphism of topological vector spaces, V is the direct product k^I of a family of copies of k, each given with the discrete topology.*

(iv) *If V is linearly compact, then no strictly weaker or stronger topology on V makes it linearly compact. Equivalently, no linear topology strictly weaker than a linearly compact topology is Hausdorff.*

Proof. (i) "If" holds by Lemma 3(i), "only if" by Lemma 4(ii).

(ii) This follows from the fact that a submodule N of a linearly topologized module M is open if and only if it is closed and M/N is discrete, together with (i).

(iii) (a) $\Longrightarrow$ (b) holds by Lemma 4(iv), and (i) above. To get (b) $\Longrightarrow$ (c), note that if (b) holds, then continuous linear maps $V \to k$ separate points of V; so a maximal linearly independent family $(\varphi_i)_{i\in I}$ of such maps gives a one-to-one continuous map $\varphi : V \to k^I$. Moreover, the linear independence of the φ_i implies that on every finite family of coordinates, the values assumed are independent; hence $\varphi(V)$ is dense in k^I. Lemmas 3(iii) and 4(iii) now imply that φ is onto; hence by (iv) of this lemma, proved below, it is an isomorphism of topological vector spaces.

(c) $\Longrightarrow$ (a) follows from Lemma 3(iv) and (i).

(iv) If T is a linearly compact topology on V, then under any Hausdorff linear topology $T' \subseteq T$, Lemma 5 tells us that V will again be linearly compact, with the same closed subspaces. Hence the same subspaces will be closed of finite codimension, i.e., by (ii), open; so the topologies are the same. This gives the second sentence of (iv), which in view of Lemma 5 is equivalent to the first. □

For the remainder of this note, we will study algebras, assuming our base ring is a field k, though many of the arguments could be carried out for more general commutative base rings. We record the following straightforward result.

LEMMA 7. *Let $A = \varprojlim_i A_i$ be the inverse limit of an inversely directed system of k-algebras. Then the multiplication of A is continuous in the inverse limit topology.*

(A very important aspect of the theory of linearly compact vector spaces which does not come into this note is the duality between the category of such spaces and the category of discrete vector spaces; cf. [3, Proposition 24.8]. More generally, [7, (29.1)] establishes a *self*-duality for the category of *locally* linearly compact spaces, i.e., extensions of linearly compact spaces by discrete spaces.)

3. Nilpotent algebras and m-separating monomials

Let A be an algebra over a field k. If B and C are k-subspaces of A, we denote by BC the k-subspace spanned by all products bc $(b \in B, c \in C)$.

We define recursively k-subspaces $A_{(n)}$ $(n = 1, 2, \ldots)$ of A by

$$A_{(1)} = A, \qquad A_{(n+1)} = \sum_{0<m<n+1} A_{(m)} A_{(n+1-m)}. \tag{3}$$

It is easy to see by induction (without assuming A associative) that for $n > 0$, $A_{(n+1)} \subseteq A_{(n)}$. These subspaces are ideals, since $A_{(n)} A = A_{(n)} A_{(1)} \subseteq A_{(n+1)} \subseteq A_{(n)}$, and similarly $A A_{(n)} \subseteq A_{(n)}$. The algebra A is said to be *nilpotent* if $A_{(n)} = \{0\}$ for some $n \geq 1$. (Some other formulations of the condition of nilpotence, which we will not need here, are shown equivalent to this one in [2, Section 4].)

Let me now preview the proof of our main result in an easy case, that of associative algebras.

Suppose A is an inverse limit of finite-dimensional associative k-algebras, and that some finitely generated subalgebra $S \subseteq A$, say generated by $g_1, \ldots, g_r$, is dense in A under the inverse limit topology.

For any $n > 1$, every element of $S_{(n)}$ is a linear combination of monomials of lengths $N \geq n$ in the given generators, each of which may be factored $a g_{i_1} \cdots g_{i_{n-1}}$, where a is a product of $N - (n-1)$ generators. It follows that

$$S_{(n)} = \sum_{i_1, \ldots, i_{n-1} \in \{1, \ldots, r\}} S g_{i_1} \cdots g_{i_{n-1}}. \tag{4}$$

Also, S is spanned modulo $S_{(n)}$ by the finitely many monomials in $g_1, \ldots, g_r$ of lengths $< n$; hence by Lemma 4(v), A, the closure of S, is spanned modulo the closure of $S_{(n)}$ by those same monomials. In particular,

(5) The closure of (4) in A has finite codimension in A.

Now consider

$$\sum_{i_1,\ldots,i_{n-1}\in\{1,\ldots,r\}} A g_{i_1} \cdots g_{i_{n-1}}. \tag{6}$$

Since the maps $a \mapsto a g_{i_1} \cdots g_{i_{n-1}}$ are continuous, the above sum is closed in A (by Lemmas 3(iii), 4(iii) and 4(v)); and it obviously contains (4) and is contained in $A_{(n)}$. So by (5), $A_{(n)}$ contains a closed subspace of finite codimension in A; hence it is open by Lemma 6(ii).

Suppose, now, that f is a homomorphism from A to a nilpotent discrete algebra. Then $\ker f$ must contain some $A_{(n)}$, hence will be open, hence f will be continuous.

Finally, by the result from [2] mentioned in the Introduction, if A is an inverse limit of nilpotent algebras, then any finite-dimensional homomorphic image of A *is* nilpotent; so in that case, any homomorphism of A to a finite-dimensional algebra is continuous.

A key aspect of the above argument was that we were able to express the general length-N monomial ($N \geq n$) in our generators as the image of a general monomial a under one of a fixed finite set of linear operators (in this case, those of the form $a \mapsto a g_{i_1} \cdots g_{i_{n-1}}$) defined using multiplications by our generators. For an arbitrary finitely generated not-necessarily-associative algebra, no such decomposition is possible, and we will see that our main result does not apply to algebras in arbitrary varieties. What we shall show next, however, is that *if* the identities of our algebra allow us to handle, in roughly this way, elements of $S_{(2)} = SS$, then, as above, we can do the same for all $S_{(n)}$.

We need some terminology. By a *monomial*, we shall mean an expression representing a bracketed product of indeterminates. For example, $(xy)z$ and $x(yz)$ are distinct monomials. (Since our algebras are not unital, we do not allow an empty monomial. We do allow monomials involving repeated indeterminates.) The *length* of a monomial will mean its length as a string of letters, ignoring parentheses; e.g., $\text{length}((xy)z) = 3$.

Note that every monomial w of length > 1 is in a unique way a product of two monomials, $w = w'w''$. Let us define recursively the *submonomials* of a monomial w: Every monomial is a submonomial of itself, and if $w = w'w''$, then the submonomials of w other than w are the submonomials of w' and the submonomials of w''. For example, the submonomials of $(xy)z$ include xy, but not yz.

We now define a technical concept that we shall need.

DEFINITION 8. If w is a monomial and m a natural number, we shall say that w is *m-separating* if w has a submonomial of length exactly $\mathrm{length}(w) - m$.

If $n \leq N$ are natural numbers, we shall call w *$[n, N]$-separating* if it is m-separating for some $m \in [n, N] = \{n, n+1, \ldots, N\}$.

For example, note that $((x_1x_2)(x_3x_4))((x_5x_6)(x_7x_8))$ has submonomials only of lengths 8, 4, 2 and 1, hence it is m-separating only for $m = 0, 4, 6, 7$; in particular, it is not $[1, 3]$-separating. On the other hand, $(((x_1x_2)(x_3x_4))((x_5x_6)(x_7x_8)))x_9$, being of length 9 and having a submonomial of length 8, is 1-separating, hence it is $[1, 3]$-separating.

Recall that a *variety* of algebras means the class of all algebras satisfying a fixed set of identities. An identity for algebras may be written as saying that a certain linear combination of monomials evaluates to zero. Such an identity is called *homogeneous* if the monomials in question all have the same number of occurrences of each variable. (It is easy to show that any variety of algebras over an infinite field is determined by homogeneous identities. An example of a variety of algebras over a finite field which is not so determined is that of Boolean rings, regarded as algebras over $\mathbb{Z}/2\mathbb{Z}$; the identity $x^2 = x$ is not a consequence of homogeneous identities of that variety. In this note, only homogeneous identities will interest us.)

Note that the variety of *associative* algebras has identities which equate every monomial with a 1-separating monomial. From this one can obtain identities which equate every monomial of length $\geq n$ with an n-separating monomial; this fact underlies the sketch just given of the proof of our main result for associative algebras. Here is the analogous relationship for general varieties.

LEMMA 9. *Suppose $\mathbf{V}$ is a variety of algebras, and d a positive integer, such that for every monomial w of length > 1, $\mathbf{V}$ satisfies a homogeneous identity equating w with a k-linear combination of $[1, 1+d]$-separating monomials.*

Then for every positive integer n and every monomial w of length $> n$, $\mathbf{V}$ satisfies a homogeneous identity equating w with a k-linear combination of $[n, n+d]$-separating monomials.

Proof. The case $n = 1$ is the hypothesis. Given $n > 1$, assume inductively that the result is true for $n - 1$, and let w be a monomial of length $> n$.

By our inductive assumption, w is congruent modulo homogeneous identities of $\mathbf{V}$ to a linear combination of $[n-1, n-1+d]$-separating monomials. By homogeneity of the identities in question, these monomials can all be assumed to have length equal to $\mathrm{length}(w)$. Suppose w' is one of these monomials which is not $[n, n+d]$-separating. Since it *is* $[n-1, n-1+d]$-separating, it must be $(n-1)$-separating; so it has a submonomial w'' of length $\mathrm{length}(w) - (n-1)$. By our original hypothesis, this w'' is congruent modulo homogeneous identities of $\mathbf{V}$ to a linear combination of $[1, 1+d]$-separating monomials. If we substitute this expression for the submonomial w'' into w', we get an expression for the latter as a linear combination of $[1+(n-1), 1+d+(n-1)]$-separating, i.e., $[n, n+d]$-separating monomials. Doing this for each such w', we get the desired expression for w modulo the identities of $\mathbf{V}$. □

The hypothesis of the preceding lemma will be used throughout the remainder of this note, so let us give it a name.

Definition 10. We shall call a variety $\mathbf{V}$ of k-algebras $[1, 1+d]$-*separative* if every monomial of length > 1 is congruent modulo homogeneous identities of $\mathbf{V}$ to a k-linear combination of $[1, 1+d]$-separating monomials. We shall call $\mathbf{V}$ *separative* if it is $[1, 1+d]$-separative for some natural number d.

4. The main theorem

The proof of the next result, our main theorem, follows the outline sketched at the beginning of the preceding section; but we will give details, and use Lemma 9 in place of familiar properties of associativity.

Theorem 11. *Let $\mathbf{V}$ be a separative variety of k-algebras, and A a linearly compact topological algebra in $\mathbf{V}$, having a finitely generated dense subalgebra S. Then*

(i) *For all $n > 0$, the ideal $A_{(n)}$ is open in the inverse limit topology on A. Hence,*

(ii) *Every homomorphism of A into a nilpotent k-algebra B is continuous* (*with respect to the discrete topology on B*). *Hence, by* [2, Theorem 10(iii)],

(iii) *If A is an inverse limit of finite-dimensional nilpotent algebras* (*topologized by the inverse limit topology, and still having a dense finitely generated subalgebra S*), *then every homomorphism from A to a finite-dimensional k-algebra B is continuous.*

Proof. (i) Given n, we wish to prove $A_{(n)}$ open. Since $A_{(1)} = A$, we may assume $n > 1$.

By the fact that A is the closure of S, and continuity of the operations of A, we have

(7) The closure of $S_{(n)}$ contains $A_{(n)}$.

Let S be generated by $g_1, \dots, g_r$. Then $S_{(n)}$ is spanned as a k-vector space by (variously bracketed) products of these elements of lengths $\geq n$. Let us understand an "m-separating product" of $g_1, \dots, g_r$ to mean an element of S obtained by substituting one of $g_1, \dots, g_r$ for each indeterminate in an m-separating monomial. Taking a d such that $\mathbf{V}$ is $[1, 1+d]$-separative, Lemma 9 (with $n-1$ in place of the n of that lemma) tells us that

(8) $S_{(n)}$ is spanned as a k-vector space by $[n-1, n-1+d]$-separating products of $g_1, \dots, g_r$ of lengths $\geq n$.

Let $U_{n,d}$ denote the finite set of all monomials $u(x_1,\dots,x_r,y)$ of lengths $n,\dots,n+d$ in $r+1$ indeterminates $x_1,\dots,x_r,y$, in which y appears exactly once. The point of this is that if w is any $[n-1,n+d-1]$-separating monomial in $x_1,\dots,x_r$, as in (8), this means that we can choose a submonomial w' such that $\mathrm{length}(w)-\mathrm{length}(w')\in[n-1,n+d-1]$; hence, w can be written $u(x_1,\dots,x_r,w')$ for some $u\in U_{n,d}$. To each $u=u(x_1,\dots,x_r,y)\in U_{n,d}$ let us associate the k-linear map $f_u:A\to A$ taking $a\in A$ to $u(g_1,\dots,g_r,a)$. Then from (8) we conclude that

$$S_{(n)}=\sum_{u\in U_{n,d}} f_u(S). \tag{9}$$

Now the closure of the right-hand side of (9) is

$$\sum_{u\in U_{n,d}} f_u(A) \tag{10}$$

by continuity of the f_u, and Lemmas 3(iii), 4(iii) and 4(v); and this sum (10) is clearly contained in $A_{(n)}$. On the other hand, as noted in (7), the closure of the left-hand of (9) contains $A_{(n)}$. So we have equality:

(11) $A_{(n)}$ is the closure of $S_{(n)}$ in A; in particular, it is a closed subspace.

To show it is open, let C_n be the k-subspace of S spanned by all monomials of length $<n$ in $g_1,\dots,g_r$. Since $g_1,\dots,g_r$ generate S, we have

$$S=C_n+S_{(n)}. \tag{12}$$

Since C_n is finite-dimensional, it is closed by Lemmas 3(i) and 4(iii), so taking the closure of (12), we get $A=C_n+A_{(n)}$ by Lemma 4(v). Hence the closed subspace $A_{(n)}$ has finite codimension in A, so by Lemma 6(ii), it is open, giving statement (i) of our theorem.

The remaining assertions easily follow. Indeed, a homomorphism f of A into a nilpotent algebra B has nilpotent image, hence has some $A_{(n)}$ in its kernel, hence that kernel is open, so f is continuous. If A is in fact an inverse limit of nilpotent algebras, then by [2, Theorem 10(iii)], its image under any homomorphism to a finite-dimensional algebra is nilpotent, so (ii) yields (iii). □

Remark. We can formally weaken the hypothesis of (iii) above to merely say A is an inverse limit of nilpotent algebras A_i (not necessarily finite-dimensional) and has a finitely generated dense subalgebra S. For we may then replace each A_i by the image of A therein, so that A maps surjectively to each of them. Then S, being dense in A, also maps surjectively to each A_i; so if it is finitely generated, so are the A_i; but a finitely generated nilpotent algebra is finite dimensional, so we are back in the situation of (iii) as stated.

Let us note the consequence of the above theorem for homomorphisms among inverse limit algebras.

Corollary 12. *Suppose that A is the inverse limit of a system of k-algebras $(A_i)_{i\in I}$ in a separative variety* $\mathbf{V}$, *that B is an inverse limit of an arbitrary system of k-algebras $(B_j)_{j\in J}$ (in each case with connecting morphisms, which we do not show); and that A has a finitely generated dense subalgebra.*

Then if either all the A_i are finite-dimensional and all the B_j nilpotent, or all the A_i are nilpotent and all the B_j finite-dimensional, then every algebra homomorphism $A \to B$ is continuous in the inverse limit topologies on A and B.

Proof. A basis of open subspaces of B is given by the kernels of its projection maps to the B_j, so it will suffice to show that the inverse image of each of those kernels under any homomorphism $f : A \to B$ is open in A. Such an inverse image is the kernel of the composite homomorphism $A \to B \to B_j$. If the A_i are finite-dimensional and the B_j nilpotent, this composite falls under case (ii) of Theorem 11, while if the A_i are nilpotent and the B_j finite-dimensional, it falls under case (iii) (as adjusted by the above Remark). In either case, the continuity given by the theorem means that the kernel of the above composite map is open, as required. $\square$

5. Separativity of some varieties, familiar and unfamiliar

Clearly, any monomial w of length > 1 is congruent modulo the consequences of the *associative* identity to a 1-separating monomial $w'x_i$. So the variety of associative k-algebras is $[1,1]$-separative, and Theorem 11 applies to it.

The variety of Lie algebras is also $[1,1]$-separative, but the proof is less trivial. When one works out the details, one sees that the argument embraces the associative case as well.

Lemma 13. *Suppose* $\mathbf{V}$ *is the variety of associative algebras, the variety of Lie algebras, or generally, any variety of k-algebras satisfying identities modulo which each of the monomials $(xy)z$, $z(xy)$ is congruent to some linear combination of the eight monomials having x or y as "outside" factor:*

$$\begin{array}{cccc} x(yz), & x(zy), & (yz)x, & (zy)x, \\ y(xz), & y(zx), & (xz)y, & (zx)y. \end{array} \tag{13}$$

Then $\mathbf{V}$ *is* $[1,1]$*-separative.*

Proof. Let w be a monomial of length > 1 which we wish to show congruent modulo the identities of $\mathbf{V}$ to a linear combination of $[1,1]$-separating, i.e., 1-separating, monomials. Let us write $w = w'w''$, and induct on min(length(w'), length(w'')).

If that minimum is 1, then w is itself 1-separating. In the contrary case, assume without loss of generality (by the left–right symmetry of our hypothesis and conclusion) that $1 < \mathrm{length}(w') \leq \mathrm{length}(w'')$, and let us write the shorter of these factors, w', as $w_1 w_2$, and rename w'' as w_3, so that $w = (w_1 w_2) w_3$. Putting w_1, w_2 and w_3 for x, y and z in the identity involving $(xy)z$ in the hypothesis of the lemma, we see that $w = (w_1 w_2) w_3$ is congruent modulo the identities of $\mathbf{V}$ to a k-linear combination of products of w_1, w_2, w_3 in each of which w_1 or w_2 is the "outside" factor. Since w_1 and w_2 each have length $< \mathrm{length}(w')$, our inductive hypothesis is applicable to the resulting products, so we may reduce them to linear combinations of 1-separating monomials, completing the proof of the general statement of the lemma.

For $\mathbf{V}$ the variety of associative algebras, the associative identity clearly yields the stated hypothesis, while if $\mathbf{V}$ is the variety of Lie algebras, two versions of the Jacobi identity, one expanding $[[x,y],z]$ and the other $[z,[x,y]]$, together do the same. □

The case of Jordan algebras is more complicated; but when one works it out, one sees a pattern of which the preceding lemma is the $d = 0$ case, while Jordan algebras come under $d = 1$. So we may consider the preceding lemma and its proof as a warm-up for the next result, giving the general case.

We note an easy fact that we will need in the proof. Let w be a monomial of length n. If $n > 1$, we can write it as a product of two submonomials; if $n > 2$ we may write one of those two as such a product, and hence get w as a bracketed product of three submonomials; and so on. We conclude by induction that

(14) For every positive $m \leq \mathrm{length}(w)$, we can write w as a bracketed product of exactly m submonomials.

We can now prove the following criterion.

Proposition 14. *Let $\mathbf{V}$ be a variety of k-algebras, and d a natural number. Then a sufficient condition for $\mathbf{V}$ to be $[1, 1+d]$-separative is that, for every monomial u obtained by bracketing the ordered string $x_1 \cdots x_{d+2}$ of $d+2$ indeterminates, each of the monomials uz, zu in $d+3$ indeterminates $x_1, \ldots, x_{d+2}, z$ be congruent modulo the homogeneous identities of $\mathbf{V}$ to a linear combination of monomials of the form $u'u''$, in which* one *of u', u'' is a product (in some order, with some bracketing) of a* proper *nonempty subset of $x_1, \ldots, x_{d+2}$. (Thus, the other factor will be a product of z and those of the x_m not occurring in the abovementioned product.)*

In particular, the preceding lemma is the case $d = 0$ of this result.

The variety of Jordan algebras over a field of characteristic not 2 satisfies this criterion with $d = 1$.

Sketch of proof. Following the pattern of the proof of the preceding lemma, assume w is a monomial of length > 1 that we want to express as a linear combination of $[1, 1+d]$-separating monomials. If it is not already $[1, 1+d]$-separating, we write $w = w'w''$, note that $\min(\mathrm{length}(w'), \mathrm{length}(w'')) \geq d+2$, and induct on that minimum. Assuming without loss of generality that w' has that minimum length, we use (14) to write $w' = u(w_1, \ldots, w_{d+2})$, for submonomials $w_1, \ldots, w_{d+2}$ of w'. We now apply to $w = w'w'' = u(w_1, \ldots, w_{d+2})w''$ an identity of the sort described in our hypothesis for uz, putting $w_1, \ldots, w_{d+2}$ for $x_1, \ldots, x_{d+2}$, and w'' for z, and note that our inductive hypothesis applies to each monomial in the resulting expression, completing the proof of our main statement.

It is easy to see that for $d = 0$, this result is equivalent to that of the preceding lemma.

To see that the variety of Jordan algebras has the indicated property with $d = 1$, first note that modulo relabeling of x_1, x_2, x_3, and consequences of the commutative identity (satisfied by Jordan algebras), all the monomials uz and zu for which we must verify that property are congruent to $z(x_1(x_2x_3))$. Hence, we need only verify it for that monomial.

To do so, we take the Jordan identity

$$(xy)(xx) = x\big(y(xx)\big), \tag{15}$$

make the substitutions $x = x_2 + x_3 + z$ and $y = x_1$, and take the part multilinear in these four indeterminates. Up to commutativity, this has only one term with z "on the outside", namely $z(x_1(x_2x_3))$, which occurs on the right-hand side. It occurs twice, but since we are assuming $\mathrm{char}(k) \neq 2$, we can divide out by 2 (which is in fact the multiplicity of every term occurring, due to the presence of a single (xx) on each side). The resulting identity expresses $z(x_1(x_2x_3))$ in the desired form. $\square$

(In the case, we have excluded from the final statement of the proposition, where $\mathrm{char}(k) = 2$, Jordan algebras are usually defined to involve operations quadratic in one of the variables, rather than just the bilinear multiplication; hence they fall outside the scope of this note. If one wants a concept of Jordan algebra over such a k involving only the bilinear multiplication, it would be natural to include among the identities of that operation the one gotten by taking the identity in x_1, x_2, x_3, z obtained by multilinearization above, written with integer coefficients, dividing all these coefficients by 2, and then reducing modulo 2. If such a definition is used, our argument for Jordan algebras is valid without restriction on the characteristic.)

One may ask whether the condition of the above proposition is necessary as well as sufficient for the stated conclusion. It is not. Using the same general approach as in the above two proofs (but noting that in proving the final statement below, one does not have recourse to left–right symmetry), the reader should find it easy to verify the following observation.

LEMMA 15. *The variety* $\mathbf{V}$ *of* k*-algebras defined by the identity*

$$(x_1x_2)(x_3x_4)=0 \tag{16}$$

is $[1,1]$*-separative.*

More generally, this is true of any variety $\mathbf{V}$ *such that modulo homogeneous identities of* $\mathbf{V}$*, the monomial* $(x_1x_2)(x_3x_4)$ *is congruent to a linear combination of monomials of the forms* ux_3 *and* ux_4*.*

6. Counterexamples

Let us now construct a k-algebra A which is an inverse limit of finite-dimensional k-algebras, and has a finitely generated subalgebra S dense in the pro-discrete topology, but which does not lie in a separative variety, and for which the conclusions of Theorem 11 fail.

We need to arrange that for some n, $S_{(n)}$ is *not* the sum of the images of finitely many *linear* polynomial operations on S. Hence we need to build up, out of a finite generating set for S, an infinite family of monomials whose sums of products will create this "problem" in $S_{(n)}$. The smallest number of generators that might work is 1, and the smallest n is 2. It turns out that we can attain these values.

Starting with a single generator p, let $p^2=q_1$, and recursively define $pq_m=q_{m+1}$. Our S will be spanned by p, these elements q_m, and the products $q_mq_n=r_{mn}$. Letting all products other than these be 0, we can describe S abstractly as having a k-basis of elements

$$p, \qquad q_m, \qquad r_{mn} \quad (m,n\geq 1), \tag{17}$$

and multiplication

(18) $pp=q_1$, $pq_m=q_{m+1}$, $q_mq_n=r_{mn}$, with all other products of basis elements 0.

For every $i>0$, S has a finite-dimensional nilpotent homomorphic image S_i defined by setting to zero all q_m with $m\geq i$ and all r_{mn} with $\max(m,n)\geq i$. Let A be the inverse limit of the system $\cdots\to S_{i+1}\to S_i\to\cdots\to S_1$. This consists of all formal infinite linear combinations of the elements (17), with multiplication still formally determined by (18).

Now if we multiply two elements $a,b\in A$, the array of coefficients of the various r_{mn} in the product, arranged in an infinite matrix, will clearly be given by the product of the column formed by the coefficients of the q's in the element a, and the row formed by the coefficients of the q's in the element b. Hence it will have rank ≤ 1, where we define the rank of an infinite matrix as the supremum of the ranks of its finite submatrices. In a linear combination of d such products, the corresponding matrix of coefficients may have rank as large as d, but we see that in no element of $A_{(2)}$ will it have infinite rank.

The set of $a\in A$ such that the matrix of coefficients in a of the r_{mn} has finite rank forms a proper k-subspace of A; e.g., it does not contain the "diagonal" element $\sum_m r_{mm}$. Thus (given the Axiom of Choice) there is a nonzero linear functional $\varphi\colon A \to k$ annihilating that subspace. Let $k\varepsilon$ denote a 1-dimensional k-algebra with zero multiplication (i.e., let $\varepsilon^2 = 0$), and define a k-linear map $f : A \to k\varepsilon$ by $f(a) = \varphi(a)\varepsilon$. From the fact that $\varphi(A_{(2)}) = \{0\}$ and the relation $\varepsilon^2 = 0$, we see that f is an algebra homomorphism. Since $\ker f$ contains all finite linear combinations of the p, q_m and r_{mn}, it has all of A as closure. But it is not all of A, so f is not continuous.

By Theorem 11, this A cannot lie in a separative variety of k-algebras. However, it does lie in the variety determined by the rather strong identities

$$((x_1x_2)x_3)x_4 = 0, \qquad x_4((x_1x_2)x_3) = 0. \tag{19}$$

Indeed, substituting any elements of A for x_1 and x_2, we find that x_1x_2 yields a formal k-linear combination of q's and r's only. Multiplying this on the right by an arbitrary element gives a formal linear combination of r's only; and this annihilates everything on both the right and the left. (Contrast Lemma 15.)

We summarize this construction as follows.

Example 16. *Let A be the linearly compact k-algebra of all formal infinite linear combinations of basis elements* (17), *with multiplication determined by* (18), *and S the subalgebra of A generated by p. Then S is dense in A, and A is an inverse limit of finite-dimensional nilpotent homomorphic images S_i of S, and satisfies the identities* (19); *but A admits a discontinuous homomorphism f to the 1-dimensional square-zero k-algebra $k\varepsilon$.*

We can get an example A^{comm} with similar properties, but with commutative multiplication, if we modify the description of the above algebra A by supplementing each relation $pq_m = q_{m+1}$ with the relation $q_mp = q_{m+1}$, and taking r_{mn} and r_{nm} to be alternative symbols for the same basis element, for all m and n. (If $\mathrm{char}(k) \neq 2$, A^{comm} can be obtained from the algebra A of the above example by using the symmetrized multiplication $x * y = xy + yx$, and passing to the closed subalgebra of A generated by p under that operation.) In this algebra, the matrix gotten by starting with the matrix of coefficients of the r_{mn} (now a symmetric matrix), and doubling the entries on the main diagonal, will have rank ≤ 2 for any product ab, so on every element of $(A^{\mathrm{comm}})_{(2)}$, its rank will again be finite. We deduce as before that this algebra admits a discontinuous homomorphism to $k\varepsilon$; we also note that it satisfies the identities

$$x_1x_2 = x_2x_1, \qquad ((x_1x_2)(x_3x_4))x_5 = 0. \tag{20}$$

We can likewise get a version A^{alt} of our construction that satisfies the alternating identity $x^2 = 0$, again by an easy modification of the algebra of Example 16, or (this time without any restriction on the characteristic), by taking an appropriate closed subalgebra of that algebra under the operation $x * y = xy - yx$. In this case, we can't have a relation $p^2 = q_1$; rather, S^{alt}

is generated by the two elements p and q_1. We find that A^{alt} satisfies the identities

$$x_1^2 = 0, \qquad \big((x_1x_2)(x_3x_4)\big)x_5 = 0. \tag{21}$$

We end this section with an easier sort of example, showing the need for the assumption in Theorem 11 that A have a finitely generated dense subalgebra. Let $k\varepsilon$ again denote the 1-dimensional zero-multiplication algebra.

Example 17. *For any infinite set I, the zero-multiplication algebra $(k\varepsilon)^I$ (which is the inverse limit of the finite-dimensional zero-multiplication algebras $(k\varepsilon)^{I_0}$ as I_0 runs over the finite subsets of I, and is trivially associative, Lie, etc.) admits discontinuous homomorphisms to the 1-dimensional zero-multiplication k-algebra $k\varepsilon$.*

Proof. Clearly any linear map between zero-multiplication algebras is an algebra homomorphism; and there exist discontinuous linear maps $(k\varepsilon)^I \to k\varepsilon$. For example, since there is no continuous linear extension of the partial homomorphism taking every element of finite support to the sum of its nonzero components, any linear extension of that map will be discontinuous. □

In contrast, if we consider algebra homomorphisms f such that the image algebra $f(A)$ has nonzero multiplication, then there are strong restrictions on examples in which, as above, the domain of f is a direct product algebra. Namely, it is shown in [4, Theorem 19], [5, Theorem 9(iii)] that if k is an infinite field, and f a surjective homomorphism from a direct product $A = \prod_I A_i$ of k-algebras to a finite-dimensional k-algebra B, and if $\mathrm{card}(I)$ is less than all uncountable measurable cardinals (a condition that is vacuous if no such cardinals exist), then writing $Z(B) = \{b \in B \mid bB = Bb = \{0\}\}$, the composite homomorphism $A \to B \to B/Z(B)$ is always continuous in the product topology on A (though the given homomorphism $f : A \to B$ may not be).

7. Some questions

The result quoted above suggests the following question.

Question 18. If k is an infinite field, and A an inverse limit of k-algebras $(A_i)_{i\in I}$ such that the indexing partially ordered set I has cardinality less than any uncountable measurable cardinal, can one obtain results like Theorem 11(ii) and (iii) for the composite map $A \to B \to B/Z(B)$ if $f : A \to B$ is surjective, without the requirement that A have a finitely generated dense subalgebra S, and/or without the hypothesis that it lie in separative variety?

A different (if less interesting) way to achieve continuity, if A does not have a dense finitely generated subalgebra, might be to refine the topology in which we try to prove our maps continuous. The topology on A defined in the next question is such that a linear map is continuous under it if and only if its restrictions to all "topologically finitely generated" subalgebras $A' \subseteq A$ are continuous in the restriction of the inverse limit topology. The question asks whether this topology is a reasonable one.

QUESTION 19. Suppose A is an inverse limit of finite-dimensional algebras, and we define a new linear topology on A by taking for the open subspaces those subspaces U whose intersections with the closures A' of all finitely generated subalgebras S' of A are relatively open in A' under the pro-discrete topology on A.

Will the multiplication of A be continuous in this topology?

For $A = (k\varepsilon)^I$ as in Example 17, the topology described above is the discrete topology on A, so in that case the answer is affirmative.

Recall next that in each of the results of Section 5, separativity was obtained from some finite family of identities. We may ask whether this is a general phenomenon.

QUESTION 20. Suppose $\mathbf{V}$ is a separative variety of k-algebras. Will some overvariety $\mathbf{V}'$ determined by finitely many identities still be separative?

If this is so, will there be such a $\mathbf{V}'$ which is $[1, 1+d]$-separative for the least d for which $\mathbf{V}$ has that property?

In Example 16, the fact that $A_{(2)}$ was not closed in A was related to the fact that it consisted of sums of arbitrarily large numbers of elements of

$$\{ab \mid a, b \in A\}. \tag{22}$$

One may ask whether the set (22) (not itself a vector subspace) is nevertheless closed in our topology on A (though a positive answer would not lead to any obvious improvement of our results). Let us generalize this question.

QUESTION 21. If A, B and C are linearly compact vector spaces, and $f : A \times B \to C$ is a continuous bilinear map, must $\{f(ab) \mid a \in A, b \in B\}$ be closed in C?

(Examination of the algebra of Example 16 shows that for that map, the answer is yes. What this says is that one can test whether an element of A has the form ab by looking at its coordinates finitely many at a time.)

We saw in Lemma 4(iv) that every linearly compact vector space is an inverse limit of finite-dimensional discrete vector spaces. Is every linearly compact algebra (i.e., every linearly compact vector space made an algebra using a continuous multiplication) an inverse limit of finite-dimensional algebras? For associative algebras—yes; in general—no! Indeed, it is not true for Lie algebras [3, Example 25.49].

In the case of Corollary 12 where A is assumed an inverse limit of finite-dimensional algebras, it is nevertheless easy to see from Theorem 11(iii) that that assumption could be weakened to make A any linearly compact algebra with a finitely generated dense subalgebra. But I don't know what happens if, in the case where B is assumed pro-finite-dimensional, we attempt the corresponding weakening.

QUESTION 22. Does the case of Corollary 12 where A is assumed an inverse limit of nilpotent algebras and B an inverse limit of finite-dimensional algebras remain true if B is merely assumed a linearly compact algebra?

Equivalently, in Theorem 11(iii), can the assumption that B is finite-dimensional be weakened to say that it is a linearly compact topological algebra?

Recall also that Corollary 12 has the peculiar hypothesis that either the A_i are finite-dimensional and the B_j nilpotent, or the A_i are nilpotent and the B_j finite-dimensional. Of the two other possible ways of distributing "finite-dimensional" and "nilpotent" among the A_i and the B_j, the arrangement that puts both conditions on the A_j and no such condition on the B_i certainly does not imply continuity; for one can take a nondiscrete A arising in this way, and let B be the same algebra with the discrete topology, regarded as an inverse limit in a trivial way. But I do not know about the reverse arrangement.

QUESTION 23. If in the last sentence of Corollary 12 we instead assume that the B_j are finite-dimensional and nilpotent (with no such condition on the A_i), can we still conclude that every algebra homomorphism $A \to B$ is continuous?

Everything we have done so far has depended on pro-nilpotence; but we may ask the following question.

QUESTION 24. Is the analog of Theorem 11(iii) true with nilpotence either replaced by other conditions (e.g., solvability, some version of semisimplicity, etc.), or dropped altogether?

The generalization of Serre's result on pro-p groups analogous to the result asked for above, i.e., with "pro-p" generalized to "profinite," has, in fact, been proved [10], [11]. The proof of this deep result uses the Classification Theorem for finite simple groups.

In connection with Question 24, let us recall briefly the meaning of solvability for a general k-algebra A; it is the straightforward extension of the condition of that name arising in the theory of Lie algebras [13, p. 17]: One defines subspaces $A^{(n)}$ $(n = 0, 1, \ldots)$ of A recursively by

$$A^{(0)} = A, \qquad A^{(n+1)} = A^{(n)} A^{(n)}, \tag{23}$$

and calls A solvable if $A^{(n)} = \{0\}$ for some n.

A difference between nilpotence and solvability which may be relevant to the above question is that a finitely generated solvable algebra, unlike a finitely generated nilpotent algebra, can be infinite-dimensional. (E.g., the S of Example 16 is solvable. There exist similar examples among Lie algebras.) So it might be necessary to make finite codimensionality of the $S^{(n)}$ in S an additional hypothesis in a version of that theorem for solvable algebras, if such a result is true.

8. Questions on subalgebras of finite codimension

The referee has raised the following interesting question.

Question 25. In cases where we have proved or conjectured that all ideals of finite codimension in an algebra A must be open (Theorem 11(iii) and Question 24), can one show more generally that all *subalgebras* of finite codimension in A are open?

While the condition that an ideal I be open means that the homomorphism $A \to A/I$ is continuous with respect to the discrete topology on A/I, the condition that a subalgebra be open has no similar interpretation. However, since a basis of open subspaces of $A = \varprojlim A_i$ is given by the kernels of the projection maps $A \to A_i$, a subalgebra will be open if and only if it contains one of these open ideals; equivalently, if and only if it is the inverse image in A of a subalgebra of one of the A_i.

Mekei [9] shows that in an *associative* algebra A, any subalgebra of finite codimension n contains an ideal of finite codimension $\leq n(n^2+2n+2)$ in A. Hence for associative A, we can indeed get results of the sort asked for: Theorem 11(iii) and Mekei's result together imply that in a topologically finitely generated inverse limit of finite-dimensional nilpotent associative algebras, every subalgebra of finite codimension is open. Positive results on Question 24 would likewise yield further results of this sort in the associative case.

Riley and Tasić [12, Lemma 2.1] prove for p-Lie (a.k.a. restricted Lie) algebras (which lie outside the scope of this note) a result analogous to Mekei's (though with an exponential bound in place of $n(n^2+2n+2)$). But as they note in [12, Example 2.2], the corresponding statement fails for ordinary Lie algebras over a field k of characteristic 0: The Lie algebra of derivations of $k[x]$ spanned by the operators $x^n d/dx$ $(n \geq 0)$ has a subalgebra B of codimension 1, spanned by those operators with $n>0$; but A is simple, so B cannot contain an ideal of A of finite codimension.

This example can be completed to a linearly compact one: In the Lie algebra A of derivations on the formal power series algebra $k[[x]]$ given by the operators $p(x)d/dx$ $(p(x) \in k[[x]])$, the operators such that $p(x)$ has constant term 0 again form a subalgebra B of codimension 1; but again, A is simple. An example not limited to characteristic 0 can be obtained from [3, Example 25.49]. There, we don't get simplicity, but still have too few ideals for there to be one of finite codimension in a certain B.

These examples show that we do not have a result like Mekei's in the variety of Lie algebras over k, and hence cannot obtain positive answers to cases of Question 25 for that variety in the way we did for associative algebras. But this does not say that such results don't hold. Indeed, the above examples used Lie algebras with a paucity of open ideals of finite codimension, while a pro-finite-dimensional algebra necessarily has a neighborhood basis of the identity consisting of such ideals. So the answer to Question 25, even for Lie algebras, remains elusive.

It would also be of interest to know for what varieties an analog of Mekei's result does hold.

QUESTION 26 (A. Mekei, personal communication). For what varieties $\mathbf{V}$ of k-algebras is it true that any subalgebra B of finite codimension in an algebra $A \in \mathbf{V}$ contains an ideal of A of finite codimension?

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