Comments, corrections, and related references welcomed, as always!

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SOME FRUSTRATING QUESTIONS ON DIMENSIONS OF PRODUCTS OF POSETS

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Abstract. The definition of the dimension of a poset is recalled. For a subposet P of a direct product of $d > 0$ chains, and an integer $n > 0$, a condition is developed which implies that for any family of n chains $(T_j)_{j\in n}$, one has $\dim(P \times \prod_{j\in n} T_j) \leq d$. Applications are noted.

Open questions, old and new, on dimensions of product posets are stated, and some other numerical invariants of posets that seem useful for studying these questions are developed. Some variants of the concept of the dimension of a poset from the literature are recalled.

In a final section, independent of the other results, it is noted that by the compactness theorem of first-order logic, an infinite poset P has finite dimension d if and only if d is the supremum of the dimensions of its finite subposets.

1. Definitions and examples

We assume the reader familiar with the definition of a *partially ordered set*, for which we will use the short term *poset*, and of the special case of a *totally ordered set*, also called a *chain*. We will formally consider a poset P to be an ordered pair $(|P|, \leq_P)$, where $|P|$ is the underlying set and $\leq_P$ the order relation; but when there no danger of ambiguity, we shall write $p \in P$ to mean $p \in |P|$, and $\leq$ for $\leq_P$. We will follow standard notational conventions such as letting $x \geq y$ denote $y \leq x$, $<$ denote the conjunction of $\leq$ with $\neq$, and $\not\leq$ the negation of $\leq$.

Posets will always be understood to be nonempty.

Recall that a set-map $f : P \to Q$ between posets is called *isotone* if

(1.1) $\quad p \leq p' \implies f(p) \leq f(p')$ for all $p, p' \in P$,

and an *embedding* if it is isotone and also satisfies

(1.2) $\quad p \not\leq p' \implies f(p) \not\leq f(p')$ for all $p, p' \in P$.

Clearly, an embedding of posets is one-to-one; but a one-to-one isotone map need not be an embedding.

A *linearization* of a partial ordering $\leq$ on a set X means a *total* ordering $\leq'$ on X which extends $\leq$, in the sense that for $x, y \in X$, if $x \leq y$ then $x \leq' y$; in other words, such that the identity map of X is an isotone map $(X, \leq) \to (X, \leq')$. It is easy to verify that every partial ordering on a set admits a linearization, and in fact that

(1.3) Every partial ordering $\leq$ on a set X is (as a set of ordered pairs) the intersection of its linearizations (Szpilrajn's Theorem [22]).

(Idea of proof: Given $x, y \in X$ such that $x \not\leq y$, show that there is a strengthening $\leq'$ of $\leq$ such that $x >' y$. Iterating this process infinitely many times if necessary, we get a linearization of $\leq$. Moreover, by our choice of the pair with $x \not\leq y$ that we start with, we can insure that any prechosen order-relation that does not hold under $\leq$ fails to hold in our linearization. Hence we have (1.3).)

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By the *product* $P \times Q$ of two posets P and Q one understands the poset whose underlying set is $|P| \times |Q|$, ordered so that $(p,q) \leq (p',q')$ if and only if $p \leq p'$ and $q \leq q'$; and analogously for products of larger families of posets. (These are in fact products in the category of posets and isotone maps, though we shall not use category-theoretic language here.) Thus (1.3) says that every poset P embeds in the product of the chains obtained from P using linearizations of $\leq_P$. Occasionally, I shall refer to a product of posets as their 'direct product', when this seems desirable for clarity.

We now come to the concept we will be studying in this note, due to B. Dushnik and E. W. Miller [7]:

Definition 1.1. *The* dimension $\dim(P)$ *of a poset* P *is the least cardinal* κ *such that*

(i) *the relation* $\leq_P$ *is an intersection of* κ *total orderings on* $|P|$,

equivalently,

(ii) P *is embeddable in a product of* κ *totally ordered sets.*

In the above definition, the implication (i) $\implies$ (ii) is clear. To get (ii) $\implies$ (i) (= [20, Theorem 10.4.2]) first consider any isotone map $f : P \to T$ where T is a totally ordered set. For each $t \in T$, regard the inverse image of t in P as a subposet, and choose a linearization $\leq_t$ of the partial ordering of that subposet of P. We can now strengthen $\leq_P$ to a linear ordering $\leq'$ of $|P|$ by making $p \leq' q$ if either $f(p) <_T f(q)$, or $f(p) = f(q)$ and $p \leq_{f(p)} q$. Thus, given an embedding of P in a product of κ posets T_α, if we construct from each of the projections $f_\alpha : P \to T_\alpha$ a linearization $\leq'_\alpha$ of $\leq_P$ as above, we see that the intersection of these linearizations will again be the partial ordering $\leq_P$.

We understand the product of the empty family of sets to be a singleton. Thus a poset has dimension 0 if and only if its underlying set is a singleton.

In the literature on dimensions of posets, condition (i) above is generally the preferred definition; but here we will more often use (ii).

This note will focus almost entirely on *finite-dimensional* posets, though we will allow underlying sets of these posets to be infinite. In indexing finite families of maps etc., we will follow the set-theorists' convention

(1.4) For n a natural number, $n = \{0, \ldots, n-1\}$.

For positive integers n, two important examples of posets of dimension n are:

(1.5) The n-cube $\mathbf{2}^n$, i.e., the n-fold direct product $\mathbf{2} \times \cdots \times \mathbf{2}$, where $\mathbf{2}$ denotes the poset $\{0,1\}$ with the ordering $0 < 1$,

and

(1.6) For $n \geq 2$, the "standard example" S_n, whose underlying set consists of $2n$ elements $\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$, with the ordering that makes each a_i less than every b_j *other* than b_i, and with no other order-relations among these $2n$ elements.

For $n \geq 3$, S_n can be identified with a subposet of $\mathbf{2}^n$, by identifying each a_i with the n-tuple that has value 1 in the i-th coordinate and 0 elsewhere, and each b_i with the n-tuple that has value 0 in the i-th coordinate and 1 elsewhere. (S_n is sometimes called the "crown" of dimension n. This nicely fits the appearance of the diagram when $n = 3$, but not so nicely for higher n, or $n = 2$.)

To see that (1.5) and (1.6) both have dimension n, note first, for $n \geq 3$, the inequalities

(1.7) $\dim(S_n) \leq \dim(2^n) \leq n$,

which are clear in view of the representation of S_n as a subposet of $\mathbf{2}^n$, and of $\mathbf{2}^n$ as a direct product of n chains. So to get the desired equalities when $n \geq 3$, it will suffice to show that the ordering on S_n is not an intersection of fewer than n total orderings.

To see this, consider any family of total orderings of $|S_n|$ whose intersection is the ordering of S_n. Note that for each $i \in n$, since $a_i \not\leq b_i$ in S_n, that family must have at least one member $\leq_i$ such that $a_i >_i b_i$. I claim these orderings $\leq_i$ must be distinct. Indeed, if for some $i \neq j$, $\leq_i$ is the same as $\leq_j$, let us call their common value $\leq_{i,j}$. Interchanging the roles of i and j if necessary, we may assume $a_j >_{i,j} a_i$. Then $a_j >_{i,j} a_i >_{i,j} b_i$, hence under the intersection of our family of orderings we have $a_j \not\leq b_i$, contradicting the definition of S_n. So the orderings $\leq_i$ $(i \in n)$ in our family are indeed distinct, so $\dim(S_n) \geq n$, so by (1.7), both (1.5) and (1.6) are n-dimensional.

As for the cases where $n < 3$: if $n = 2$ it is clear that both $\dim(S_2)$ and $\dim(\mathbf{2}^2)$ are > 1, and the latter is ≤ 2 by definition. With the help of the embedding of S_2 in $\mathbf{3}^2$ as $\{(0,1),(1,0),(1,2),(2,1)\}$, we see that its dimension is also ≤ 2; so both are indeed equal to 2. For $n = 1$, S_1 is undefined, and $\dim(\mathbf{2}^1) = 1$ is clear.

For another class of examples, recall that a poset P in which every pair of distinct elements is incomparable is called an *antichain*. We claim that

(1.8) Every antichain P with more than one element has dimension 2.

Indeed, if one chooses any total ordering of $|P|$, then the intersection of that ordering and the opposite ordering is the antichain ordering, so $\dim(P) \leq 2$; and since $|P|$ has more than one element, no single linear ordering makes it an antichain.

We remark that under an isotone bijection of posets which is *not* an isomorphism, the dimension may increase, decrease, or remain unchanged. For instance, starting with an antichain of 8 points, a bijection onto the poset $\mathbf{2}^3$ is an isotone map that increases the dimension from 2 to 3, while a linearization decreases the latter dimension from 3 to 1. On the other hand, a bijection from a 4-element antichain to the poset $\mathbf{2}^2$ is isotone and leaves the dimension, 2, unchanged.

Let us note one more elementary poset construction (a generalization of the one we applied to chains in proving the equivalence of the two conditions in Definition 1.1). Given a poset P, and for each $p \in P$ a poset Q_p, we may define the set

(1.9) $|\sum_P Q_p| \ = \ \{(p,q) \mid p \in P,\ q \in Q_p\},$

and partially order this set lexicographically; that is, by defining

(1.10) $(p,q) \ \leq \ (p',q')$ if and only if either $p <_P p'$, or $p = p'$ and $q \leq_{Q_p} q'$.

It is not hard to verify that for this construction,

(1.11) $\dim(\sum_P Q_p)$ is the supremum of $\{\dim(P)\} \cup \{\dim(Q_p) \mid p \in P\}$.

In the verification, one takes a representation of P as in Definition 1.1(i), and representations of the posets Q_p as in Definition 1.1(ii) (which may, but need not, have the more restricted form of condition (i)). By repeating mappings if necessary, one can assume that the number of chains used in each of these representations is the supremum indicated above. One can then easily combine these to get an embedding of $\sum_P Q_p$ in a product of that number of chains. This gives "$\leq$" in (1.11); "$\geq$" is straightforward.

2. Our main result, and some consequences

To lead up to our main result, let us first note that for any two posets P and Q (which, we recall, are required to be nonempty), we clearly have

(2.1) $\max(\dim(P), \dim(Q)) \ \leq \ \dim(P \times Q) \ \leq \ \dim(P) + \dim(Q).$

Though the properties of vector-space dimension suggest that the second inequality should be equality, the two concepts of dimension are not alike in that respect. To see an interesting way that equality can fail, consider a poset P of dimension d, represented as a subposet of a product of d chains, $P \subseteq \prod_{i \in d} T_i$, and consider a nontrivial chain C. Suppose that for each i we let $T_i^* = T_i \ltimes C$, the product-set ordered lexicographically – intuitively, the chain gotten from T_i by replacing each element t by a miniature copy of C. It is easy to see that the map $f : P \times C \to \prod_{i \in d} T_i^*$ taking $((p_0, \dots, p_{d-1}), c)$ to $((p_0, c), \dots, (p_{d-1}, c))$ is one-to-one and isotone. Can it fail to be an embedding?

In general, yes. If $p < p'$ in P and $c > c'$ in C, then in the product poset $P \times C$, the elements (p,c) and (p',c') are by definition incomparable; but if it happens that for all i, we have $p_i < p'_i$, then under the ordering described above, $f(p,c) < f(p',c')$; so their images are not incomparable.

However, *if* the subposet $P \subseteq \prod_{i \in d} T_i$ has the property that whenever two elements p, p' satisfy $p < p'$, then as members of $\prod_{i \in d} T_i$ they agree in *at least one* coordinate, then looking at that coordinate, we see that we do get incomparability between the indicated elements of $f(P \times C)$; so f is an embedding, so we indeed have $\dim(P \times C) = d = \dim(P)$.

The condition that every pair of comparable elements of $P \subseteq \prod_{i \in d} T_i$ agree in at least one coordinate may seem unnatural, but for $d \geq 3$ it is easy to see that it holds in the poset S_d, regarded as a subposet of $\mathbf{2}^d$ (sentence following (1.6)).

In fact, more is true in that example: Every pair of elements $a_i < b_j$ agrees in one coordinate where their common value is 0 (the j-th) and one where their common value is 1 (the i-th); and the above construction can be adapted to show, as a consequence, that taking the product of S_d with *two* chains does not increase its dimension.

The next result gives this argument in detail for a more general situation, that applies to products of a poset P with possibly more than two chains. Note, however, that in that result, d is not assumed to be the dimension of P (as was the case in the above example), but simply to be an integer such that P is a subposet of a product of d chains with certain properties.

Theorem 2.1. *Let d and n be nonnegative integers, $(T_i)_{i\in d}$ a d-tuple of chains, which for notational convenience we will assume pairwise disjoint, P a subposet of $\prod_{i\in d} T_i$, and $(M_j)_{j\in n}$ an n-tuple of pairwise disjoint subsets of $\bigcup_{i\in d} T_i$, such that*

(2.2) *For every comparable pair of elements $p < p'$ in P, and every $j \in n$, there exists $i \in d$ such that the i-th coordinates of p and p' are equal, and their common value is a member of M_j.*

Then for any n-tuple of chains $(C_j)_{j\in n}$ we have

(2.3) $\dim(P \times \prod_{j\in n} C_j) \ \leq\ d.$

Proof. For notational simplicity, we may assume all the C_j are equal to a common chain C, and that $\bigcup_{j\in n} M_j = \bigcup_{i\in d} T_i$. Indeed, given structures as in the statement of the theorem, we may embed all the C_j in a common chain C, and if we prove (2.3) with the C_j all replaced by C, this will imply the same inequality for the original choices of C_j. Likewise, if we enlarge some of the M_j (still keeping them disjoint) so that their union becomes the whole set $\bigcup_{i\in d} T_i$, then the hypothesis about pairs of elements $p < p'$ assumed for the original choices of M_j remains true. So let us make those assumptions.

To obtain (2.3), we need an embedding of the poset on the left-hand side, which we can now write $P \times C^n$, in a product of d chains T'_i $(i \in d)$. For this purpose we define

(2.4) $T'_i \ =\ T_i \ltimes C$, the set-theoretic product of T_i and C ordered lexicographically, i.e., so that

(2.5) $(t, c) \ \leq\ (t', c')$ if and only if $t < t'$, or $t = t'$ and $c \leq c'$.

Also, since the M_j $(j \in n)$ partition $\bigcup_{i\in d} T_i$, we can set the notation

(2.6) For each $t \in \bigcup_{i\in d} T_i$, $m(t) \in n$ will denote the unique value such that $t \in M_{m(t)}$.

We now define a map $f : P \times C^n \to \prod_{i\in d} T'_i$ as follows.

(2.7) For $p = (p_i)_{i\in d} \in P \subseteq \prod_{i\in d} T_i$, and $c = (c_j)_{j\in n} \in C^n$, let $f(p, c)$ be the element of $\prod_{i\in d} T'_i$ whose i-th coordinate is $(p_i, c_{m(p_i)})$ for each $i \in d$.

We wish to show that f is an embedding of posets.

First, f is isotone, i.e.,

(2.8) If $(p, c) \leq (p', c')$ in $P \times C^n$, then $f(p, c) \leq f(p', c')$ in $\prod_{i\in d} T'_i$.

This follows easily from (2.5) and (2.7).

To complete the proof that f is an embedding, we need to show that

(2.9) If $(p, c) \not\leq (p', c')$, then $f(p, c) \not\leq f(p', c')$.

This breaks down into two cases. Suppose first that

(2.10) $p \ \not\leq\ p'$.

In that case, for some i we have $p_i \not\leq p'_i$, i.e., $p_i > p'_i$, and looking at the i-th coordinates of $f(p, c)$ and $f(p', c')$, namely $(p_i, c_{m(p_i)})$ and $(p'_i, c'_{m(p'_i)})$, we see from the lexicographic ordering of T'_i that the former is $>$ the latter, giving the conclusion of (2.9).

If, on the other hand,

(2.11) $p \leq p'$, but $c \not\leq c'$,

then in view of the second of these conditions, we may choose a $j \in n$ such that

(2.12) $c_j \ >\ c'_j$.

Now by the *first* condition of (2.11) and the hypothesis (2.2), there is some $i \in d$ such that

(2.13) $p_i \ =\ p'_i \in M_j$.

Note that by (2.7),

(2.14) $f(p,c)$ and $f(p',c')$ have i-th terms (p_i, c_j) and (p'_i, c'_j) respectively.

Since the T_i-coordinates of the above two i-th terms are the same by (2.13), the order-relation between them is that of the C-coordinates, which satisfy (2.12). This completes the proof of (2.9), and of the Theorem. □

I came up with the above result after pondering [5], which showed by an explicit construction that $\dim(S_3 \times \mathbf{2} \times \mathbf{2}) = 3$, i.e., is equal to $\dim(S_3)$. The next corollary includes that case.

Corollary 2.2. *Suppose $d \geq 2$ and $(T_i)_{i\in d}$ is a family of non-singleton chains, each having a least element 0_i and a greatest element 1_i, and P is a subposet of $\prod_{i\in d} T_i$ consisting of elements each of which has at least one coordinate of the form 0_i and at least one of the form $1_{i'}$. Then for any two chains C_0 and C_1, we have*

(2.15) $\dim(P \times C_0 \times C_1) \ \leq \ d.$

In particular, if $d \geq 3$ and P is any subposet of $\mathbf{2}^d \setminus \{0,1\}$, containing the standard poset S_d (e.g., if $P = S_d$), then for any two chains C_0 and C_1,

(2.16) $\dim(P \times C_0 \times C_1) \ = \ d \ = \ \dim(P).$

Proof. In the context of the first assertion, given $p \leq p' \in P$, since p has at least one coordinate of the form 1_i, $p' \geq p$ must also have 1_i as its i-th coordinate; and similarly, since p' has a coordinate $0_{i'}$, p must agree with p' in that coordinate. Applying Theorem 2.1 with $n = 2$, $M_0 = \{0_i \mid i \in d\}$ and $M_1 = \{1_i \mid i \in d\}$, we get (2.15).

The second statement follows because when each T_i is the 2-element set $\{0_i,\, 1_i\}$, the exclusion of 0 and 1 from P forces every element to have at least one coordinate of the form 1_i and one of the form $0_{i'}$; and the assumption that P contains S_d, which we saw following (1.6) has dimension d, turns the inequality (2.15) into equality. □

If, instead of assuming that every element has at least one 0-coordinate and at least one 1-coordinate, we only assume one of these conditions, it is easy to check that the analogous reasoning gives a conclusion half as strong:

Corollary 2.3. *Suppose $d \geq 1$, $(T_i)_{i\in d}$ is a family of chains each having a least element, 0_i, and P is a subposet of $\prod_{i\in d} T_i$ consisting of elements each of which has at least one coordinate of the form 0_i. Then for any chain C we have*

(2.17) $\dim(P \times C) \ \leq \ d.$

In particular, if P is any subposet of $\mathbf{2}^d \setminus \{1\}$ containing S_d (e.g., if $P = S_d \cup \{0\}$) then for any chain C we have

(2.18) $\dim(P \times C) \ = \ d \ = \ \dim(P).$

The analogous statements hold with least elements 0_i and 0 everywhere replaced by greatest elements 1_i and 1. □

To get examples of Theorem 2.1 with $n > 2$, we shall use a different way of choosing n subsets M_j of $\bigcup_{i\in d} T_i$, based on the subscript i rather than the distinction between greatest and least elements of T_i. We will need a bit of notation.

(2.19) For every positive integer d and every subset $A \subseteq d{+}1 = \{0, \ldots, d\}$, P_d^A will denote the subset of $\mathbf{2}^d$ consisting of those elements in which the number of coordinates of the form 1_i is a member of A.

For instance, when $d \geq 3$, $S_d = P_d^{\{1,d-1\}}$.

Then we have

Corollary 2.4. *For every positive integer d, every integer a with $0 \leq a \leq d-1$, and every family of chains $(C_j)_{j\in n}$ with*

(2.20) $n \ \leq \ d/2,$

we have, in the notation of (2.19),

$$(2.21) \qquad \dim(P_d^{\{a,\,a+1\}} \times \prod_{j\in n} C_j) \ \leq \ d.$$

Proof. Again writing $\mathbf{2}^d$ as $\prod_{i\in d}\{0_i, 1_i\}$, let us define n subsets of $\bigcup_{i\in d}\{0_i, 1_i\}$ by

$$(2.22) \qquad M_j \ = \ \{0_{2j}, 1_{2j}, 0_{2j+1}, 1_{2j+1}\}, \ \ (j = 0, \ldots, n-1).$$

Condition (2.20) guarantees that these formulas indeed describe subsets of $\bigcup_{i\in d}\{0_i, 1_i\}$.

Now if $p \leq p'$ are elements of $P_d^{\{a,\,a+1\}}$, then the cardinalities of the subsets of d on which p and p' assume values of the form 1_i (namely, a or $a+1$), differ by at most 1, hence the order relation between them necessitates that they disagree at most one coordinate. Since each M_j contains both 0_i and 1_i for *two* values of i, one of those values of i must have the property that p and p' agree on the i-th coordinate. Hence these subsets M_j satisfy the hypotheses of Theorem 2.1, completing the proof. □

If the posets $P_d^{\{a,\,a+1\}} \subseteq \mathbf{2}^d$ had, like S_d, dimension d, then the above result would give us, for every n, finite posets whose dimensions were not changed on taking a direct product with n chains. However, such subposets of $\mathbf{2}^d$ generally have dimension less than d. There are many results in the literature obtaining bounds on the dimensions of subposets of $\mathbf{2}^d$; cf. [6], [9], [12], [14, Theorem 7.1], [16], [17, Theorems 7, 10, 12 and 13], and [23, Theorem 2]. Let us obtain one such result here, as another consequence of Theorem 2.1. (In the statement and proof, we shall not need to look at the disjoint union of the factors of $\mathbf{2}^d$, hence we shall not, as in the preceding corollaries, treat these as disjoint sets $\{0_i, 1_i\}$, but as the same set $\mathbf{2} = \{0, 1\}$.)

Corollary 2.5. *Let d be a positive integer, and let*

$$(2.23) \qquad P \ = \ P_d^{[2,\,d-2]} \setminus \{(0, \ldots, 0, 1, 1), \ (1, \ldots, 1, 0, 0)\}.$$

(*Here $P_d^{[2,\,d-2]}$ is defined as in* (2.19)*, with $A = [2,\, d-2] = \{2, 3, \ldots, d-2\}$; $(0, \ldots, 0, 1, 1)$ denotes the d-tuple whose first $d-2$ coordinates are 0 and whose last two coordinates are 1, and $(1, \ldots, 1, 0, 0)$ is the corresponding d-tuple with the roles of 0 and 1 reversed.*) *Then*

$$(2.24) \qquad \dim(P) \ \leq \ d-2.$$

Hence the same upper bound holds for the dimension of any subposet of P; in particular, for any poset of the form $P_d^{\{a,\,b\}}$ with $3 \leq a < b \leq d-3$.

Proof. I claim that as a subset of $\mathbf{2}^d$,

$$(2.25) \qquad P \ \subseteq \ P_{d-2}^{[1,\,d-3]} \times \mathbf{2} \times \mathbf{2};$$

in other words, that the first $d-2$ coordinates of every $p \in P$ contain at least one 1 and at least one 0. To see the former condition, note that being contained in $P_d^{[2,\,d-2]}$, p itself must have at least two coordinates 1. If none of these were among the first $d-2$ coordinates, this would force $p = (0, \ldots, 0, 1, 1)$, but that element is excluded in (2.23). The second condition follows in the same way.

Now by Corollary 2.2, with $d-2$ in place of d, and each T_i taken to be $\mathbf{2}$, we see that the right-hand side of (2.25) has dimension $\leq d-2$, hence the same is true of its subposet P, proving (2.24)

The final sentence of the corollary follows immediately. □

3. An old result, and questions old and new

Having seen that dimension of posets is *not* in general additive on direct products, it is striking that in an important class of cases, it is. The theorem below was proved, for products over index sets of arbitrary cardinality, in K. Baker's unpublished undergraduate thesis [1]. That proof is summarized in [14] for pairwise products (from which the case of arbitrary *finite* products follows), assuming the dimensions finite. I give below a version of the proof (also for pairwise products of finite-dimensional posets) which I find easier to follow.

Theorem 3.1. ([1, p.9, Property 3], [14, p.179, last 11 lines]) *Let P and Q be finite-dimensional posets, each having a least element 0 and a greatest element 1. Then*

$$(3.1) \qquad \dim(P \times Q) \ = \ \dim(P) + \dim(Q).$$

Proof. In view of Definition 1.1(ii), $\leq$ clearly holds in (3.1), so it suffices to show $\geq$. For this we shall use Definition 1.1(i), and show that if we have an expression for the partial order of $P \times Q$ as an intersection of

(3.2) $\quad n$ total orderings, $\leq_0, \ldots, \leq_{n-1}$, on $|P \times Q|$,

then we can split that family of orderings into two disjoint subsets, such that a certain map of P into the product of chains determined by one of those subsets is an embedding, as is a map of Q into the product determined by the other. Thus the former subset must consist of $\geq \dim(P)$ orderings and the latter of $\geq \dim(Q)$, giving $n \geq \dim(P) + \dim(Q)$, as desired.

The trick to finding this partition is to look at the relative order, under each of the orderings (3.2), of the elements $(0_P, 1_Q)$ and $(1_P, 0_Q)$ of $|P \times Q|$. Reindexing those n orderings if necessary, we may assume that for some $m \leq n$,

(3.3) $\quad (0_P, 1_Q) <_i (1_P, 0_Q)$ for $0 \leq i < m$, while $(1_P, 0_Q) <_i (0_P, 1_Q)$ for $m \leq i < n$.

Let us now show that the map

(3.4) $\quad P \to (|P \times Q|, \leq_0) \times \cdots \times (|P \times Q|, \leq_{m-1})$ given by $p \mapsto ((p, 0_Q), \ldots, (p, 0_Q))$,

which is clearly isotone, is an embedding; i.e., that if $p,\, p' \in P$ satisfy

(3.5) $\quad p \not\leq p'$,

then the corresponding condition holds on the images of p and p' under (3.4).

To get this, note that under the product ordering on $P \times Q$, the relation (3.5) implies that $(p, 0_Q) \not\leq (p', 1_Q)$. Hence we can find some i such that

(3.6) $\quad (p, 0_Q) \not\leq_i (p', 1_Q)$.

Now if $i \geq m$, we would have $(p, 0_Q) \leq_i (1_P, 0_Q) <_i (0_P, 1_Q) \leq_i (p', 1_Q)$, contradicting (3.6); so $i < m$. Since (3.6) implies $(p, 0_Q) \not\leq_i (p', 0_Q)$, this completes the proof that (3.4) respects $\not\leq$, hence is an embedding. The analogous argument shows that the map $Q \to (|P \times Q|, \leq_m) \times \cdots \times (|P \times Q|, \leq_{n-1})$ given by $q \mapsto ((0_P, q), \ldots, (0_P, q))$ likewise gives an embedding of Q. As noted in the first paragraph of this proof, this establishes (3.1). $\square$

Though we have seen that (3.1) does not hold without the assumptions that P and Q have upper and lower bounds, the deviations from equality in all known examples are ≤ 2, leading to the longstanding open question:

Question 3.2. (i) *If P and Q are finite-dimensional posets, must* $\dim(P \times Q) \geq \dim(P) + \dim(Q) - 2$ *?*

In particular,

(ii) *If P is a finite-dimensional poset, and n a positive integer such that* $\dim(P \times \mathbf{2}^n) = \dim(P)$, *must* $n \leq \mathbf{2}$ *?*

In [14, last line of p.179 and top line of p.180] it was conjectured that $\dim(P) + \dim(Q)$ can exceed $\dim(P \times Q)$ by at most the number of members of the set $\{P,\, Q\}$ that do *not* have both a greatest and a least element. Thus, the case where that number is 0 is Theorem 3.1, while the case where there is no restriction on P or Q corresponds to Question 3.2(i) above. The case of that conjecture where that number is 1, however, turned out to be false: from Corollary 2.2 we see that for any $n \geq 3$, $\dim(S_n \times \mathbf{2}^2) = \dim(S_n)$, so $\dim(S_n) + \dim(\mathbf{2}^2)$ exceeds $\dim(S_n \times \mathbf{2}^2) = \dim(S_n)$ by 2, though $\mathbf{2}^2$ has both greatest and least elements.

However, one might ask about different intermediate cases between those of Theorem 3.1 and Question 3.2(i), suggested by Corollary 2.3, which we note as Question 3.3 below. C. Lin poses part (iii) of that question in [17], though Theorem 10 and Lemma 11 of that paper suggest the stronger implications of (i) and (ii).

Question 3.3. (i) *If P and Q are finite-dimensional posets such that P has a least element and Q has a greatest element, must* $\dim(P \times Q) \geq \dim(P) + \dim(Q) - 1$ *?*

(ii) *If P and Q are finite-dimensional posets each of which has a least element, or each of which has a greatest element, must* $\dim(P \times Q) \geq \dim(P) + \dim(Q) - 1$ *?*

And finally, a possible implication weaker than either of the above two:

(iii) [17, p.80] *If P and Q are finite-dimensional posets such that P has a least or a greatest element, and Q has both, must* $\dim(P \times Q) \geq \dim(P) + \dim(Q) - 1$ *?*

Turning back to Question 3.2, I wondered whether the result of Theorem 3.1, where no dimensionality was lost in forming a product of posets, could be related to the hypothesis that the sets of minimal and of maximal elements of each factor, being singletons, had dimension 0. If so, could one use the fact that for arbitrary finite P and Q the sets of minimal and maximal elements are antichains, hence have dimension ≤ 2, to get a positive answer to Question 3.2 for finite posets? But I see no way to adapt the proof of Theorem 3.1.

Moving on to other questions, recall that in Theorem 2.1, the result did not depend on the lengths of the chains C_j. This suggests

Question 3.4. (*Cf.* [18], [21, Conjecture 1]) *If P is a finite-dimensional poset, and C any chain of more than one element, must* $\dim(P \times C) = \dim(P \times \mathbf{2})$ *?*

The next question at first seemed "obviously" to have an affirmative answer; but I see no argument proving it.

Question 3.5. *If P is a finite-dimensional poset, and* $\dim(P \times \mathbf{2}) = \dim(P) + 1$, *must* $\dim(P \times \mathbf{2} \times \mathbf{2}) = \dim(P) + 2$ *?*

Much more generally (but much more vaguely), we may ask

Question 3.6. *Given finite-dimensional posets P_0, P_1, P_2, if we know their dimensions, and those of $P_0 \times P_1$, $P_0 \times P_2$ and $P_1 \times P_2$, what can we say about* $\dim(P_0 \times P_1 \times P_2)$ *? (Anything more than that it is greater than or equal to the dimensions of each of the pairwise products, and less than or equal to the values of* $\dim(P_i \times P_j) + \dim(P_k)$ *for* $\{i, j, k\} = \{0, 1, 2\}$ *?)*

One example of "misbehavior" on this front: Suppose P is an antichain of more than one element, and $P' = P \times \mathbf{2}$. Now P, P^2, and P^3 are all antichains, hence all have dimension 2; but P', P'^2, and P'^3 have the forms $P \times \mathbf{2}$, $P^2 \times \mathbf{2}^2$, and $P^3 \times \mathbf{2}^3$, and from (1.11) we see that they have dimensions 2, 2, and 3 respectively. So the dimensions of three posets and of their pairwise direct products do not determine the dimension of the product of all three.

4. Absorbency

The following concept might be helpful in studying questions of the sort we have been considering.

Definition 4.1. *If P is a finite-dimensional poset, let us define its* absorbency *to be*

(4.1) $\mathrm{abs}(P)$ = *the largest natural number n such that* $\dim(P \times \prod_{i \in n} T_i) = \dim(P)$ *for every n-tuple of chains* $(T_i)_{i \in n}$.

As an example,

(4.2) For all $d \geq 3$, $\mathrm{abs}(S_d) = 2$.

Here $\geq$ follows from the $P = S_d$ case of (2.16). To get the reverse inequality, we will use a result from the literature. Note that if $\mathrm{abs}(S_d)$ were ≥ 3, then since $S_d \subseteq \mathbf{2}^d$, the value of $\dim(S_d \times S_d)$ would be $\leq \dim(S_d \times \mathbf{2}^d) \leq \dim(S_d \times \mathbf{2}^3) + \dim(\mathbf{2}^{d-3}) = d + (d-3) = 2d - 3$. However, Trotter shows in [23, Theorem 2] that for all $d \geq 3$, $\dim(S_d \times S_d) = 2d - 2$.

An immediate property of absorbency is

(4.3) $\mathrm{abs}(P) \leq \dim(P)$,

since for any $n > \dim(P)$, the product of P with, say, $\mathbf{2}^n$ will contain a copy of $\mathbf{2}^n$, hence have dimension greater than $\dim(P)$.

Here are some other easy properties.

Lemma 4.2. *Let P and Q be finite-dimensional posets.*

(4.4) *If* $\mathrm{abs}(P) \geq \dim(Q)$, *then* $\dim(P \times Q) = \dim(P)$, *and* $\mathrm{abs}(P \times Q) \geq \mathrm{abs}(P) - \dim(Q) + \mathrm{abs}(Q)$.

The situation where neither the hypothesis of (4.4), *nor the corresponding inequality with the roles of P and Q reversed holds is covered by*

(4.5) *If* $\dim(Q) \geq \mathrm{abs}(P)$ *and* $\dim(P) \geq \mathrm{abs}(Q)$, *then*
$\max(\dim(P), \dim(Q)) \leq \dim(P \times Q) \leq \dim(P) + \dim(Q) - \max(\mathrm{abs}(P), \mathrm{abs}(Q))$.

Proof. In (4.4), the first conclusion holds because Q embeds in a product of $\dim(Q) \leq \mathrm{abs}(P)$ chains, which P can "absorb" without increasing its dimension.

To get the final inequality of (4.4), we must show that the dimension of $P \times Q$ is not increased on multiplying it by a product of $\mathrm{abs}(P) - \dim(Q) + \mathrm{abs}(Q)$ chains. Let us write such a product as $X \times Y$, where X is a product of $\mathrm{abs}(P) - \dim(Q)$ chains, and Y a product of $\mathrm{abs}(Q)$ chains. Then if we write $P \times Q \times (X \times Y)$ as $P \times (X \times (Q \times Y))$, we see that $Q \times Y$ has dimension $\dim(Q)$ (by our choice of Y), hence $X \times (Q \times Y)$ has (by our choice of X) dimension at most $(\mathrm{abs}(P) - \dim(Q)) + \dim(Q) = \mathrm{abs}(P)$, so its product with P has dimension $\dim(P)$, which we have noted equals $\dim(P \times Q)$, so we have indeed shown that the dimension of $P \times Q$ has not been increased, as required.

In the conclusion of (4.5), the left-hand inequality is immediate. The right-hand inequality is equivalent to saying that $\dim(P \times Q)$ is bounded above by both $\dim(P) + \dim(Q) - \mathrm{abs}(P)$ and $\dim(P) + \dim(Q) - \mathrm{abs}(Q)$, so by symmetry it suffices to prove the former bound. If we embed Q in a product of $\dim(Q)$ chains, and break this into a product X of $\mathrm{abs}(P)$ chains and a product Y of $\dim(Q) - \mathrm{abs}(P)$ chains, then we have $\dim(P \times Q) \leq \dim(P \times X \times Y) \leq \dim(P \times X) + \dim(Y) = \dim(P) + (\dim(Q) - \mathrm{abs}(P))$, as desired. □

We may ask

Question 4.3. *Under the hypothesis of* (4.4)*, does equality always hold in the final inequality of that implication?*

If we ask the same question about the final inequality of (4.5), the peculiarities noted in the final paragraph of §3 make trouble. (Using (1.11), one can show that for P' as defined in that paragraph, the absorbency of P' is 1, and deduce that if in (4.5) we take both P and Q to be the poset there called P', that final inequality becomes $2 < 3$.) But those peculiarities involved disconnected posets, and I know of no cases not of that sort. Recall that a poset P is called disconnected if it can be written as a union of subposets P_0 and P_1 such that all elements of P_0 are incomparable with all elements of P_1, and connected if it cannot be so written. We can thus ask

Question 4.4. *If the posets P and Q in* (4.5) *are connected, must equality hold in the final inequality of the conclusion thereof?*

Since (1.11) gives us a way of computing the dimension of a disconnected poset from the dimensions of its connected components, a positive answer to Question 4.4 (together with the first assertion of (4.4)) would allow us to compute exactly the dimension of a product of two arbitrary finite-dimensional posets, given the dimensions and absorbencies of their connected components. It would, in particular, imply positive answers to Questions 3.4 and 3.5.

But as long as we do not know a positive answer to Question 4.4, here are some further points worth considering:

If Question 3.4 has a negative answer, one could consider variants of the absorbency concept, depending on the lengths of the chains involved.

If Question 3.4 has a positive answer but Question 3.5 does not, one might define the "eventual absorbency" of a finite-dimensional poset P as the supremum, as $n \to \infty$, of $(\dim(P) + n) - \dim(P \times \mathbf{2}^n)$.

Given a poset P and a nonnegative integer d about which we know that $\dim(P) \leq d$, one might define the absorbency of P *relative* to d to be the largest n such that the product of P with every n-tuple of chains continues to have dimension $\leq d$. So, for instance, given finite-dimensional posets P and Q, the absorbency of $P \times Q$ relative to $\dim(P) + \dim(Q)$ will be at least the sum of the absorbencies of P and of Q.

5. Bounded dimension

Let us call a poset *bounded* if it has a greatest element and a least element.

(I wish I knew a better term for this condition, since it has very different properties from the familiar sense of "bounded". E.g., a subposet of a bounded poset is not, in general, bounded. However, "bounded" is used in this way in places in the literature; e.g., [14, p.179, 12th line from bottom].)

This allows us to define another function which is useful in studying dimensions:

Definition 5.1. *For P a finite-dimensional poset, the* bounded dimension *of P, denoted* bd-dim(P), *will be defined to be*

(i) *the greatest integer* n *such that* P *contains a bounded subposet* P' *satisfying* $\dim(P') = n$, *equivalently,*

(ii) *the maximum, over all pairs of elements* $p \leq p'$ *in* P, *of* $\dim(\{x \mid p \leq x \leq p'\})$.

Since every bounded subset P' of P is contained in an interval $\{x \mid p \leq x \leq p'\}$, we see that the two versions of the above definition are equivalent. Because of Theorem 3.1, this function behaves very nicely under direct products:

Lemma 5.2. *For finite-dimensional posets* P *and* Q,

(5.1) $\quad \text{bd-dim}(P \times Q) \ = \ \text{bd-dim}(P) \ + \ \text{bd-dim}(Q).$

Proof. For elements $p \leq p'$ of P, let us write

(5.2) $\quad [p, p'] \ = \ \{p'' \in P \mid p \ \leq \ p'' \ \leq \ p'\}$, regarded as a subposet of P.

Thus, by version (ii) of the definition of bounded dimension, $\text{bd-dim}(P \times Q)$ is the greatest of the values $\dim([(p,q), (p',q')])$ for $(p,q) \leq (p',q')$ in $P \times Q$.

But by the order-structure of a direct product, $[(p,q), (p',q')]$ is isomorphic to $[p, p'] \times [q, q']$, and by Theorem 3.1 the dimension of this product is $\dim([p,p']) + \dim([q,q'])$. Taking the maximum over all pairs $p \leq p'$ and $q \leq q'$, we get $\text{bd-dim}(P) + \text{bd-dim}(Q)$. $\square$

Clearly, for any poset P,

(5.3) $\quad \text{bd-dim}(P) \ \leq \ \dim(P).$

However, the two sides of (5.3) can be far from equal, as can be seen by taking $P = S_d$ for large d. Then the left-hand side of (5.3) is easily seen to be 1, while the right-hand side is d.

The concept of bounded dimension allows us to get an upper bound on absorbency:

(5.4) $\quad \text{abs}(P) \ \leq \ \dim(P) \ - \ \text{bd-dim}(P).$

Indeed, if we take the product of P with more nontrivial chains than the number on the right, then since P contains a bounded set of dimension $\text{bd-dim}(P)$, and each of the chains we multiply by contains a nontrivial bounded chain, the whole product will have dimension $> \text{bd-dim}(P) + (\dim(P) - \text{bd-dim}(P)) = \dim(P)$. Thus, to be "absorbed", a family of nontrivial chains can have at most the number of members shown on the right in (5.4). (Incidentally, this argument is easily adapted to give the same upper bound for the possibly larger "eventual absorbency" function defined in the next-to-last paragraph of the preceding section.)

The two sides of (5.4) can also be far from equal. For instance, for $P = S_d$, where $d \geq 3$, we have seen in (4.2) that the left-hand side of (5.4) is 2. On the other hand, the right-hand side is $d - 1$.

From (5.4) we get a strengthening of (4.3); namely, we can add to that inequality the observation,

(5.5) The only finite-dimensional posets P for which $\text{abs}(P) = \dim(P)$ are the antichains,

since a poset that is not an antichain contains a copy of $\mathbf{2}$, and so has bounded dimension at least 1.

Trotter [23, Conjecture 2] suggested that for all $n \geq 2$ there exist posets P of dimension n such that $P \times P$ also has dimension n. Reuter [21, Theorem 13] showed that no such P exists for $n = 3$; but it is conceivable that there exist such P for higher n. Such an example would imply a negative answer to Question 4.4, in view of (5.4).

It might also be of interest to look at the variants of the bounded-dimension function in which *bounded* is replaced by *bounded above* or *bounded below.*

6. Boolean dimension

Another variant of the dimension function, studied, inter alia, in [2], [4], [10], [19], [25], is based on the concept of a *Boolean representation* of a poset P. Here one considers families of $d > 0$ total orderings $<_0, \ldots, <_{d-1}$ of $|P|$, which are *not* assumed to be strengthenings of the ordering $<_P$ of P, but merely to have the property that whether a pair of elements $x \neq y \in P$ satisfies $x <_P y$ is a function of the set $\{i \in d \mid x <_i y\}$. For example, the order relation of S_n $(n \geq 3)$ can be so described in terms of the four total orderings

(6.1) $\quad a_0 <_0 \cdots <_0 a_i <_0 \cdots <_0 a_{n-1} <_0 b_0 <_0 \cdots <_0 b_i <_0 \cdots <_0 b_{n-1},$

(6.2) $\quad b_0 <_1 \cdots <_1 b_i <_1 \cdots <_1 b_{n-1} <_1 a_0 <_1 \cdots <_1 a_i <_1 \cdots <_1 a_{n-1},$

$$a_0 <_2 b_0 <_2 \cdots <_2 a_i <_2 b_i <_2 \cdots <_2 a_{n-1} <_2 b_{n-1}, \tag{6.3}$$

$$b_0 <_3 a_0 <_3 \cdots <_3 b_i <_3 a_i <_3 \cdots <_3 b_{n-1} <_3 a_{n-1}. \tag{6.4}$$

Namely, $x < y$ in S_n if and only if on the one hand, x precedes y in (6.1) but follows it in (6.2) (which together say that x has the form a_i and y the form b_j), and, further, the relative order of x and y is the same in (6.3) as in (6.4) (which says that (x, y) is *not* a pair of the form (a_i, b_i)). This is called a *Boolean representation* of S_n in terms of the four total orderings (6.1)-(6.4). The *Boolean dimension* of a poset P, $\mathrm{bdim}(P)$, is the least d such that P has a Boolean representation in terms of d total orderings. This number is always $\leq \dim(P)$, but often strictly less; e.g., the above example shows that for all n, $\mathrm{bdim}(S_n) \leq 4$.

In the formal definition of a Boolean representation of $<_P$ in terms of total orderings $<_0, \ldots, <_{d-1}$, one begins by mapping $\{(x, y) \in |P| \times |P| \mid x \neq y\}$ to $\mathbf{2}^d$ by sending (x, y) to the d-tuple which has 1 in the i-th position if and only if $x <_i y$. The ordering of P is then determined by a function $\tau : \mathbf{2}^d \to \{0, 1\}$, such that $x <_P y$ if and only if τ takes the d-tuple determined by (x, y) to 1. For instance, in the above example describing S_n, τ is the function on $\mathbf{2}^4$ taking the value 1 only at the two 4-tuples $(1, 0, 0, 0)$ and $(1, 0, 1, 1)$. Here the condition that for a 4-tuple to be taken to 1, its first entry must be 1 and its second entry 0 translate the conditions that x must precede y in (6.1) but not in (6.2); and the condition that the last two entries of the 4-tuple be equal translates the condition that x and y must occur in the same order in (6.3) and in (6.4). The term "Boolean dimension" refers to the fact that a map $\tau : \mathbf{2}^d \to 2$ can be expressed by a Boolean word in d variables.

In [25] it is shown that $\mathrm{bdim}(P)$ agrees with $\dim(P)$ when the latter is 3. On the other hand, [4, Lemma 6] shows that $\mathrm{bdim}(\mathbf{2}^6) = 5$, whence $\mathrm{bdim}(\mathbf{2}^d) \leq \lceil 5d/6 \rceil$ for all d [4, Theorem 5]. However, [4, Proposition 9] shows that for each d, the d-th power of a large enough finite chain has Boolean dimension d. In [4, Lemma 4] the inequality $\mathrm{bdim}(P \times Q) \leq \mathrm{bdim}(P) + \mathrm{bdim}(Q)$ is proved.

A complication is that there are two versions of the definition of Boolean dimension in the literature, which *almost* agree, but not quite. Continuing to write $\mathrm{bdim}(P)$ for the dimension function defined as above, the other, which I shall denote $\text{leq-bdim}(P)$, is defined as the least positive integer d such that there exists a d-tuple of linear orderings of $|P|$ with the property that for *all* pairs (x, y) of elements of P (no longer required to be distinct), the condition $x \leq_P y$ can be expressed as a Boolean function of the d conditions $x \leq_i y \ \ (i \in d)$.

It is easy to see that $\mathrm{bdim}(P) \leq \text{leq-bdim}(P)$: if the condition $\leq_P$ on arbitrary pairs can be expressed as a Boolean function of the d conditions $x \leq_i y$, then restricting that Boolean function to pairs of distinct elements $x \neq y$, the conditions $x \leq_i y$ become $x <_i y$, and we get a Boolean realization of $<_P$ in terms of these relations.

The reverse implication is less obvious, since if we are given a Boolean realization of $<_P$ in terms of d linear orderings $<_i$, that Boolean function applied to the relations $\leq_i$ might not give 1 on ordered pairs of the form (x, x).

However, that reverse implication does in fact hold, except when P is a non-singleton antichain. I do not know of any statement of this result in the literature. (I am told that workers in the field are aware of there being two slightly different versions of Boolean dimension; but this is rarely mentioned in print. The one point I am aware of where it is mentioned is in [2], footnotes to pp.245 and 248.) So let us prove this almost-equivalence result below.

The proof will use the observation that given any d-tuple of total orderings $<_i \ \ (i \in d)$ of a set $|P|$, in terms of which a partial ordering $<_P$ can be described on pairs of distinct elements by a Boolean expression, one can replace any subset of the orderings $<_i$ by the opposite orderings, and still express $<_P$ in terms of the resulting linear orders; this is done simply by inserting appropriate "not" operators in the Boolean expression for $<_P$. (The corresponding statement is not true for expressions of $\leq_P$ in terms of the $\leq_i$, since on pairs (x, y) not required to satisfy $x \neq y$, the relation $x \geq_i y$ is not the negation of $x \leq_i y$.)

Lemma 6.1. *For all finite posets P other than non-singleton antichains,* $\mathrm{bdim}(P) = \text{leq-bdim}(P)$.

On the other hand, if P is a non-singleton antichain, then $\mathrm{bdim}(P) = 1$, *while* $\text{leq-bdim}(P) = 2$.

Proof. For the first assertion, we have noted that $\mathrm{bdim}(P) \leq \text{leq-bdim}(P)$, so what we must prove is the reverse inequality. A singleton poset can easily be seen to have both dimensions 1, so assume P is a poset that is not an antichain, and that $<_P$ can be written as a Boolean expression in linear orderings $<_0, \dots, <_{d-1}$ on $|P|$.

Since P is not an antichain, the Boolean expression in question must assume the value 1 on some string of 0's and 1's. If we replace the orderings corresponding to the 0's (if any) in that string with the opposite orderings, we get, as discussed in the paragraph before the statement of this lemma, a realization of $<_P$ using a Boolean expression which now takes the value 1 on $(1, \dots, 1)$.

If we now apply this Boolean expression to the relations $\leq_0, \dots, \leq_{d-1}$, the result will behave on pairs with distinct entries like our realization of $<_P$, while on pairs (x, x), since these satisfy $x \leq_i x$ for all i and our Boolean expression maps $(1, \dots, 1)$ to 1, it will also agree with $\leq_P$. This proves the first assertion of the lemma.

To get the second, let P be a non-singleton antichain. We can establish $\mathrm{bdim}(P) = 1$ by choosing any total ordering $<_0$ of $|P|$, and using as realizer the Boolean operation 0 (making elements incomparable regardless of which is larger under $<_0$). This does not work for $\text{leq-bdim}(P)$ because it would also give $x \not\leq_P x$ for $x \in |P|$; and it is equally easy to see that none of the other three Boolean operations in one variable work. However, if we take any linear ordering $\leq_0$ on $|P|$, and let $\leq_1$ be the opposite ordering, we see that the relation $(x \leq_0 y) \wedge (x \leq_1 y)$ is equivalent to $x = y$, and hence realizes the antichain ordering $\leq_P$; so $\text{leq-bdim}(P) = 2$. □

(In the definition of $\mathrm{bdim}(P)$, it is tempting to allow $d = 0$, which would come into play only when P was an antichain, in which case the element 0 of the free Boolean ring on zero generators, which is $\{0, 1\}$, would indeed determine the relation $x <_P y$ on pairs $x \neq y$ as never holding. However, under this definition, the result $\mathrm{bdim}(P \times Q) \leq \mathrm{bdim}(P) + \mathrm{bdim}(Q)$ would fail when one of P, Q was an antichain and the other was not – the proof of that inequality requires $\mathrm{bdim}(P)$ and $\mathrm{bdim}(Q)$ to be realized by Boolean expressions on nonempty families of linearizations. Whether it would be best to allow $d = 0$ in the definition of bdim, making antichains an exception to the result on Boolean dimensions of product posets, or to keep the requirement $d > 0$ in the definition of bdim, is for those in the field to decide.)

7. Other sorts of dimension in the literature

Another variant of the dimension of a poset P described in [2] is the *local dimension,* $\mathrm{ldim}(P)$, the least d such that there exists a family of linearizations of *subposets* of P such that each member of P appears in $\leq d$ of these linearized subposets, and such that for every pair (x, y) of distinct elements of P, there are enough linearized subposets containing both so that the relationship between x and y in P (an order-relation, or incomparability) is the relation between their images in the product of these linearizations. As with the Boolean dimension, this is $\leq \dim(P)$, and considerably less for the S_n (in this case always ≤ 3).

Still another variant: If instead of looking at lists of linearizations of $<_P$, such that every relation $x \not<_P y$ is realized in at least one member of our list, and letting the dimension of P be the least cardinality of such a list, one can look at lists of linearizations with a "weight", a positive real number, attached to each linearization, such that $x \not<_P y$ if and only if the sum of the weights of the linearizations for which $x > y$ is at least 1, and define the *fractional dimension* of P to be the infimum, over all weighted lists which determine the ordering of P in this way, of the sum of the weights of the listed linearizations. (The term "weight" is not used in the literature; it is my way of giving an intuitive description of the definition.) See [15, p.5] and references given there. [15] also introduces several dimension-like functions specific to the type of posets there called convex geometries.

Less exotic invariants of posets considered in [2] and elsewhere are the *height,* i.e., the supremum of the cardinalities of chains contained in P, and the *width,* the supremum of the cardinalities of antichains in P. The function associating to a poset P the largest n such that P contains a subposet isomorphic to S_n is denoted $\mathrm{se}(P)$, and studied in [3] and [24, §5.2.1].

The function $\dim(P)$, though usually (as in this note) simply called the dimension of P, is sometimes, as in the title of [2], called the *Dushnik-Miller* dimension, to distinguish it amid this plethora of variants.

8. When an infinite poset has finite dimension

I have left this topic to the end, because it assumes familiarity with a very different technique from those used in the other sections. Namely, by a straightforward application of the *Compactness Theorem* of first-order logic [8, Theorem VI.2.1(b)], [11, Theorem 6.1.1] (applied to a language with a constant for each member of $|P|$, and using version (i) of Definition 1.1), one can immediately verify

Proposition 8.1. *Let P be a poset of arbitrary cardinality, and d a positive integer. Then P has dimension d if and only if d is the supremum of the dimensions of all finite subposets of P.* □

(I originally had two rather complicated proofs of this result – one using a carefully chosen ultraproduct of the finite subposets of P, the other on a tricky transfinite recursion. But Theodore Slaman pointed out to me that the Compactness Theorem yields the same result immediately.)

Immediate consequences regarding a couple of the other invariants we have defined are

Corollary 8.2. *For any finite-dimensional poset P, its bounded dimension* $\mathrm{bd\text{-}dim}(P)$ *is equal to the supremum of the values of* $\mathrm{bd\text{-}dim}(P')$ *on finite subposets $P' \subseteq P$.* □

Corollary 8.3. *For any finite-dimensional poset P, its absorbency* $\mathrm{abs}(P)$ *is equal to the* infimum *of the values of* $\mathrm{abs}(P')$ *over the finite subposets P' of P satisfying* $\dim(P') = \dim(P)$. □

I wonder about

Question 8.4. *Do there exist* infinite *cardinals κ and λ such that for every poset P, if all subposets $P' \subseteq P$ of cardinality $< \kappa$ have dimension $\leq \lambda$, then P itself has dimension $\leq \lambda$?*

Proposition 8.1 is the corresponding statement with $\aleph_0$ in the role of κ, and the natural number n in place of the infinite cardinal λ.

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