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BILINEAR MAPS ON ARTINIAN MODULES

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Abstract. It is shown that if a bilinear map $f : A \times B \to C$ of modules over a commutative ring k is nondegenerate (i.e., if no nonzero element of A annihilates all of B, and vice versa), and A and B are Artinian, then A and B are of finite length.

Some consequences are noted. Counterexamples are given to some attempts to generalize the above result to balanced bilinear maps of bimodules over noncommutative rings, while the question is raised whether other such generalizations are true.

Below, rings and algebras are associative and unital, except where the contrary is stated.

Examples of modules over a commutative ring k that are Artinian but not Noetherian are well known; for example, the $\mathbb{Z}$-module Z_{p^∞}. However, such modules do not show up as underlying modules of k-algebras. (We shall see in §3 that this can be deduced, though not trivially, from the Hopkins-Levitzki Theorem, which says that left Artinian rings are also left Noetherian.) The result of the next section can be thought of as a generalization of this fact.

I am grateful to J. Krempa for pointing out two misstatements in the first version of this note, to K. Goodearl, D. Herbera, T. Y. Lam and L. W. Small for references to related literature, and to the referee of a previous version of this note for an alternative proof of the main theorem, noted at the end of §1.

1. Our main result

In the proof of the following theorem, it is interesting that everything before the next-to-last sentence works, mutatis mutandis, if A and B are both assumed Noetherian rather than Artinian, though the final conclusion is clearly false in that case. (On the other hand, the argument does not work at all if one of A and B is assumed Artinian, and the other Noetherian.)

Theorem 1. *Suppose k is a commutative ring, and $f : A \times B \to C$ a bilinear map of k-modules which is nondegenerate, in the sense that for every nonzero $a \in A$, the induced map $f(a, -) : B \to C$ is nonzero, and for every nonzero $b \in B$, the induced map $f(-, b) : A \to C$ is nonzero.*

Then if A and B are Artinian, they both have finite length.

Proof. If elements $a \in A$, $b \in B$ satisfy $f(a, b) = 0$, we shall say they *annihilate* one another. (The concept of an element c of the base ring k annihilating an element x of A, B, or C will retain its usual meaning,

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$c\,x = 0$.) For subsets $X \subseteq A$, respectively $Y \subseteq B$, we define the annihilator sets

$$(1) \qquad \begin{aligned} X^{\perp} &= \{b \in B \mid (\forall\, x \in X)\ f(x,b) = 0\} \subseteq B, \\ Y^{\perp} &= \{a \in A \mid (\forall\, y \in Y)\ f(a,y) = 0\} \subseteq A. \end{aligned}$$

We see that these are submodules of B and A respectively, that the set of annihilator submodules in B (respectively, in A) forms a lattice (in the order-theoretic sense) under inclusion, and that these two lattices of annihilator submodules are antiisomorphic to one another, via the maps $U \mapsto U^{\perp}$. (This situation is an example of a "Galois connection" [2, §5.5], but I will not assume familiarity with that formalism.)

When A and B are Artinian, these lattices of submodules of A and B both have descending chain condition; so since they are antiisomorphic, they also have ascending chain condition. Hence all their chains have finite length. Let us choose a maximal (i.e., unrefinable) chain of annihilator submodules of A,

$$(2) \qquad \{0\} = A_0 \subseteq A_1 \subseteq \dots \subseteq A_n = A.$$

This yields a maximal chain of annihilator submodules of B,

$$(3) \qquad B = B_0 \supseteq B_1 \supseteq \dots \supseteq B_n = \{0\},$$

where

$$(4) \qquad B_i = A_i^{\perp}, \qquad A_i = B_i^{\perp}.$$

It is easy to see that for each $i < n$, f induces a k-bilinear map

$$(5) \qquad f_i : (A_{i+1}/A_i) \times (B_i/B_{i+1}) \to C,$$

via

$$(6) \qquad f_i(a + A_i,\, b + B_{i+1}) = f(a,b) \quad (a \in A_{i+1},\ b \in B_i).$$

I claim that under f_i, every nonzero element of A_{i+1}/A_i has zero annihilator in B_i/B_{i+1}, and vice versa. For if we had a counterexample, say $a \in A_{i+1}/A_i$, then its annihilator would be a proper nonzero annihilator submodule of B_i/B_{i+1}, and by (6) this would lead to an annihilator submodule of B strictly between B_i and B_{i+1}, contradicting the maximality of the chain (3).

From this we can deduce that every nonzero element of A_{i+1}/A_i and every nonzero element of B_i/B_{i+1} have the same annihilator in k. Indeed, if $c \in k$ annihilates the nonzero element $x \in A_{i+1}/A_i$, then for every $y \in B_i/B_{i+1}$, $c\,y$ will annihilate x, hence must be zero; so every $c \in k$ annihilating one nonzero member of A annihilates all of B, and dually.

It is now easy to see that the common annihilator of all nonzero elements of these two modules is a prime ideal $P_i \subseteq k$, making k/P_i an integral domain, such that A_{i+1}/A_i and B_i/B_{i+1} are k/P_i-modules. Moreover, taking any nonzero element of either of our modules, say $x \in A_{i+1}/A_i$, we have $kx \cong k/P_i$ as modules, so since A_{i+1}/A_i is Artinian, so is k/P_i.

But an Artinian integral domain is a field; so A_{i+1}/A_i and B_i/B_{i+1} are vector spaces over the field k/P_i, so the fact that they are Artinian means that they have finite length. Thus, A and B admit finite chains (2), (3) of submodules with factor-modules of finite length; so they are each of finite length. □

Here is an illuminating alternative proof of Theorem 1, sketched by the referee of an earlier version of this note, which uses the theory of "secondary representations" of modules over a commutative ring [11, Appendix to §6]. Let k, A, B, C, f be as in the hypothesis of our theorem.

Because A is Artinian, it has, by [11, Theorem 6.11], a representation $A = A_1 + \dots + A_n$, where each A_m is a P_m-*secondary* module for some prime ideal $P_m \subseteq k$. This means that

(7) for every $c \in P_m$, some power c^i annihilates A_m,

and

(8) for every $c \in k - P_m$, we have $c\,A_m = A_m$.

Now for each m, the fact that A_m is Artinian clearly implies that there exists a finitely generated subideal $P' \subseteq P_m$ having the same annihilator in A_m that P_m has. From this one can show by induction that for every $i \geq 0$, the ideal $(P')^i$ likewise has the same annihilator in A_m as $(P_m)^i$ does [6, Lemma 3, p.54]. (Key steps: if $\{0\} = (P')^i\,x$, write this $\{0\} = P'(P')^{i-1}x$, and conclude by choice of P' that $\{0\} = P_m(P')^{i-1}x$. By commutativity of k, write this $\{0\} = (P')^{i-1}P_m\,x$. By induction on i, conclude that $\{0\} = (P_m)^{i-1} P_m\, x = (P_m)^i x$.) Since P' is finitely generated, (7) shows that some $(P')^{i_m}$ annihilates A_m, and by the above observation, this gives

$$(9) \qquad (P_m)^{i_m} A_m \ = \ \{0\}.$$

Note next that if any of the ideals $P_m \subseteq k$ is nonmaximal, then for every maximal ideal $Q \subseteq k$, we have $Q\, A_m = A_m$ by (8). But since $B/A_m^\perp$ is Artinian, it has a minimal nonzero submodule $k\, x$, and this will have for annihilator in k a maximal ideal Q; hence x annihilates $Q\, A_m = A_m$, contradicting the assumption $x \in B/A_m^\perp - \{0\}$. (This is the only step in this proof where the bilinear map f is used.) So all the ideals P_m are maximal.

Now consider, for each m, the chain $A_m \supseteq P_m\, A_m \supseteq \cdots \supseteq (P_m)^{i_m} A_m = \{0\}$, where i_m is as in (9). The factor-module at each step is a vector space over the field k/P_m, hence, being Artinian, must be finite-dimensional. So A_m has finite length; and since this holds for every m, A has finite length, as claimed.

2. Some immediate consequences

We start with a trivial consequence of our theorem.

Corollary 2. *If, in the statement of Theorem 1, we assume only one of the nondegeneracy conditions, namely that for all nonzero $a \in A$, the induced map $f(a, -) : B \to C$ is nonzero* (*respectively, that for all nonzero $b \in B$, the induced map $f(-, b) : A \to C$ is nonzero*), *we can still conclude that A* (*respectively, B*) *has finite length.*

Assuming neither nondegeneracy condition, we can still conclude that $A/B^\perp$ and $B/A^\perp$ have finite length.

Proof. Without any nondegeneracy assumption, note that f induces a *nondegenerate* bilinear map

$$(10) \qquad A/B^\perp \ \times \ B/A^\perp \ \to \ C.$$

Since $A/B^\perp$ and $B/A^\perp$ are again Artinian, we can apply Theorem 1 to (10) and conclude that both these factor-modules have finite length.

If we assume one of our nondegeneracy conditions, which we may write $A = A/B^\perp$, respectively, $B = B/A^\perp$, then by the above result we get finite length for one of A, B. □

We can now recover a result of A. Facchini, C. Faith and D. Herbera.

Corollary 3 ([4, Proposition 6.1])**.** *The tensor product $A \otimes_k B$ of two Artinian modules over a commutative ring k has finite length.*

Proof. Letting $A^\perp$ and $B^\perp$ be annihilators with respect to the tensor multiplication $\otimes : A \times B \to A \otimes_k B$, the preceding corollary tells us that $A/B^\perp$ and $B/A^\perp$ have finite length. But $A \otimes_k B$ can be regarded as the tensor product of these factor modules; and a tensor product of modules of finite length over a commutative ring has finite length. □

Let us next apply Theorem 1 to the multiplication of a k-algebra R. This does not require R to be associative, so we shall make no such assumption. In the study of nonassociative algebras, it is often not natural to require a unit; but without one, nondegeneracy of the multiplication is not automatic; so in the statement below, we get this nondegeneracy by dividing out by an appropriate annihilator ideal.

(Caveat: below, we name that ideal $Z(R)$, though it is not the center of R. This notation, from [3], is based on the phrase "zero multiplication", and also on the use of that symbol in the theory of Lie algebras, where it does coincide with the center.)

Corollary 4. *Let R be a not-necessarily-associative algebra over a commutative ring k, and let*

$$(11) \qquad Z(R) \ = \ \{x \in R \mid xR = Rx = \{0\}\}.$$

Then if $R/Z(R)$ is Artinian as a k-module, it is also Noetherian as a k-module.

Hence, if $Z(R) = \{0\}$ (*in particular, if R has a unit element*), *then if R is Artinian as a k-module, it is also Noetherian as a k-module.*

Proof. Let

(12) $\quad Z_l(R) \;=\; \{x \in R \mid xR = \{0\}\,\}, \quad Z_r(R) \;=\; \{x \in R \mid Rx = \{0\}\,\}.$

Thus, $Z(R) = Z_l(R) \cap Z_r(R)$. The multiplication of R induces a nondegenerate bilinear map of k-modules $R/Z_l(R) \times R/Z_r(R) \to R$. Since $R/Z_l(R)$ and $R/Z_r(R)$ are homomorphic images of $R/Z(R)$, they are Artinian over k, hence by Theorem 1 they are Noetherian over k. Hence $R/Z(R)$, which embeds in $R/Z_l(R) \times R/Z_r(R)$, is also Noetherian.

The assertion of the final sentence clearly follows. □

This shows, as mentioned at the start of this note, that groups such as Z_{p^∞} cannot be the additive groups of rings. There are other groups, such as $\mathbb{Q}/\mathbb{Z}$, which one feels should be excluded for similar reasons, though they are not themselves Artinian. This leads us to formulate the following consequence of Theorem 1.

Corollary 5. *Suppose k is a commutative ring, and $f : A \times B \to C$ a nondegenerate bilinear map of k-modules.*

Then if B (respectively, A) is locally Artinian (i.e., if every finitely generated submodule thereof is Artinian), then every Artinian submodule of A (respectively, B) has finite length.

Proof. Let A_0 be an Artinian submodule of A, and consider the set S of annihilator submodules in A_0 of finitely generated submodules of B. By our nondegeneracy assumption, the intersection of all members of S is $\{0\}$. But S is closed under finite intersections, so by descending chain condition on submodules of A_0, its intersection again belongs to S. Thus, $\{0\}$ is the annihilator in A_0 of some finitely generated submodule $B_0 \subseteq B$. Assuming B locally Artinian, B_0 will be Artinian, so we can apply Corollary 2 to the restricted map $f : A_0 \times B_0 \to C$, and conclude, as desired, that A_0 has finite length.

By symmetry, we have the corresponding implication with the roles of A and B interchanged. □

One has obvious analogs of Corollaries 2, 3 and 4 for this result. From the last of these, we see that the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$, which is locally Artinian but has Artinian submodules of infinite length, cannot be the additive group of a unital (associative or nonassociative) ring.

3. Relation to the Hopkins-Levitzki Theorem

Going back to Corollary 4, can the case of that result where R is an associative unital k-algebra be deduced, alternatively, from the well-known fact that a left Artinian ring is also left Noetherian – the Hopkins-Levitzki Theorem [8, Theorem 4.15(A)] ?

Well, if, as assumed in Corollary 4, R is Artinian as a k-module, then it is certainly Artinian as a left R-module, hence if it is associative and unital, the Hopkins-Levitzki Theorem says it is Noetherian as a left R-module. Can we somehow get from this that it is Noetherian as a k-module?

We can, using the following striking result.

(13) (Theorem of Lenagan and Herbera [12, Theorem on p.2044].) If R and S are rings, and ${}_R M_S$ is a bimodule which is left Noetherian and right Artinian, then M is also left Artinian and right Noetherian.

We will use a special case of the above fact:

Corollary 6 (to (13))**.** *If R is a k-algebra, and ${}_R M$ an R-module of finite length as an R-module, and if, moreover, as a k-module M is Artinian or Noetherian, then it has finite length as a k-module.*

Proof. Regard M as a bimodule ${}_R M_k$. Then (13) or its left-right dual, applied to this bimodule, gives the desired conclusion. □

So in the associative unital case of Corollary 4, once we know by the Hopkins-Levitzki Theorem that R has finite length as a left R-module, the above result gives an alternative proof of the conclusion of that corollary.

(Note on the background of (13): In [12], (13) is described as a result of T. Lenagan, with a new proof communicated to the author by D. Herbera. Lenagan [10] had indeed proved the hard part of (13), that if M is both Artinian and Noetherian on one side, and Noetherian on the other, then it is also Artinian on the latter side; and this is what is called Lenagan's Theorem in most sources, e.g, [9]. However, Herbera's version in [12] tacitly supplies the additional argument showing that if M is merely Artinian on one side and Noetherian on the other, then it is also Noetherian on the former side. It is that part, and not the part proved by Lenagan, that we needed for our alternative proof of the associative unital case of Corollary 4. Incidentally, Lenagan formulated his result for 2-sided ideals, but as noted in [9, p. 332, sentence after Theorem 11.30], his proof carries over verbatim to bimodules.)

For some results on when not necessarily unital associative right Artinian rings must be right Noetherian, and related questions, see [5] and papers cited there.

Returning to the relation between Corollary 4 and the Hopkins-Levitzki Theorem, we cannot hope to go the other way, and obtain the latter from the former: we can't get started, since the Artinian assumption on R as a left R-module does not, by itself, give such a condition on R as a k-module. This suggests that we look for some result that can be applied directly to the right and left R-*module* structures of R; say a generalization of Theorem 1 to a result on balanced bilinear maps

(14) $\quad f: {}_SA_R \times {}_RB_T \ \to \ {}_SC_T$

of bimodules over associative rings.

I have not been able to find such a generalization. In the next section we take a brief look at the situation.

4. Some counterexamples, and a question

Given a map (14), the annihilator of every element of A is a T-submodule of B, and the annihilator of every element of B is an S-submodule of A; so one might hope for a result assuming the Artinian property for ${}_SA$ and B_T. But this does not work: if we take a field k, two k-algebras S and T, any nonzero Artinian left S-module A, and an Artinian but non-Noetherian right T-module B (for instance, $S = T = k[t]$, $A = B = k((t))/k[[t]]$, equivalently, $k[t, t^{-1}]/k[t]$), then we see that the canonical map

(15) $\quad \otimes: \ {}_SA_k \times {}_kB_T \ \to \ {}_S(A \otimes_k B)_T$

is a counterexample: it is nondegenerate, and ${}_SA$ and B_T are Artinian, but they are not both Noetherian.

If, instead, we assume the Artinian condition on A_R and ${}_RB$, counterexamples are harder to find; but we can get them using the following construction.

Lemma 7. *Let k be a field, R_0 a k-algebra, and ${}_{R_0}M$ any nonzero left R_0-module. Then there exists a k-algebra R having*

(i) *a left module ${}_RB$ whose submodule lattice is isomorphic to the result of adjoining to the submodule lattice of ${}_{R_0}M$ one new element at the bottom,*

(ii) *a* simple *right module A_R, and*

(iii) *a nondegenerate R-balanced k-bilinear map $f: A_R \times {}_RB \to k$.*

Proof. From ${}_{R_0}M$, we shall construct B as a k-vector-space. We shall then define R as a certain k-algebra of linear endomorphisms of B, and A as a certain k-vector space of linear functionals $B \to k$, closed under right composition with the actions of members of R. The map $f: A \times B \to C$ will be the function that evaluates members of A at members of B. Here are the details.

Writing ω for the set of natural numbers, let $B \subseteq M^\omega$ be the vector space of those sequences $x = (x_i)_{i\in\omega}$ which have the same value at all but finitely many i. Let R be the k-algebra of k-linear maps $B \to B$ spanned by

(16) those maps which act by multiplication by an element $r \in R_0$ simultaneously on all coordinates; i.e., by $rx = (rx_i)_{i\in\omega}$,

and

(17) those maps which act by projecting to the sum of finitely many of our copies of M (regarded as a direct summand in M^ω), then mapping this sum into itself by an arbitrary finite-rank k-vector-space endomorphism.

It is easy to see that every nonzero R-submodule of B contains the submodule B_{fin} of all elements having finite support in ω, that an R-submodule B' containing B_{fin} is determined by the values that elements of B' assume "almost everywhere", and that the set of these values can be, precisely, any submodule of M. Thus, the lattice of submodules of B containing B_{fin} is isomorphic to the lattice of submodules of M; so the full lattice of submodules of B consists of this and a new bottom element, the zero submodule. This gives (i).

Let A be the set of all k-linear functionals a on B that depend on only finitely many coordinates (i.e., for which there exists a finite subset $I \subseteq \omega$ such that a factors through the projection of $B \subseteq M^{\omega}$ to M^I). This set is easily seen to be closed under right composition with elements of R; hence we may regard A as a right R-module, and evaluation of elements of A on elements of B gives an R-balanced k-bilinear map $f : A \times B \to k$. Further, for any $a \in A - \{0\}$ and $b \in B - \{0\}$, we can clearly find a $u \in R$ of the sort described in (17) which carries b to an element not in $\ker(a)$, so that $0 \neq f(a, ub) = f(au, b)$. In particular, f has the two properties defining nondegeneracy (statement of Theorem 1), giving (iii).

It remains to show that A_R is simple, as required by (ii). Given $a \in A - \{0\}$ and $a' \in A$, we see that there will exist $y \in B$ with finite support such that $a(y) = 1$. Choosing such a y, define $u : B \to B$ by $u(x) = a'(x)y$. It is easy to check that u has the form (17), hence lies in R, and that it satisfies $au = a'$, proving simplicity. □

Taking for ${}_{R_0}M$ in the above lemma any Artinian non-Noetherian module over a k-algebra R_0, and letting $S = T = k$, ${}_SC_T = {}_kk_k$, we see that f is a nondegenerate balanced bilinear map (14) with A_R and ${}_RB$ Artinian (and A_R Noetherian), but with ${}_RB$ non-Noetherian.

However, neither the examples obtained using (15) nor those gotten as above satisfy all four possible Artinian conditions on A and B. So we ask

Question 8. *If (14) is a nondegenerate balanced bilinear map, and if all of ${}_SA$, A_R, ${}_RB$ and B_T are Artinian, must these modules also be Noetherian?*

If the answer is positive, one could look at intermediate cases, e.g., where three of the above Artinian conditions are assumed. (The case of (15) where $S = k$, ${}_SA = {}_kk$, and B_T is an Artinian non-Noetherian right module over a k-algebra T, shows that the assumption that all the above modules *except* ${}_RB$ are Artinian is *not* sufficient to prove ${}_RB$ or B_T Noetherian; though it does not say whether those conditions are sufficient to make ${}_SA$ and/or A_R Noetherian.)

Some results with the desired sort of conclusion, but with hypotheses of a stronger sort than those suggested above, are proved in [12].

Alongside Question 8 and its close relatives, one might look for results with the weaker conclusion that A or B have ascending chain condition on sub*bimodules.*

Since we noted an alternative proof of Theorem 1 using the theory of secondary representations of modules, we might ask whether there is anything analogous to that theory in the noncommutative context. Results in that direction are obtained in [1], but I don't see a way of using them to answer Question 8. (Straightforward analogs of the results we applied are not true; and we also called on commutativity of k at a key step.)

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