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Two Statements about Infinite Products that Are Not Quite True

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To Don Passman, on his 65th birthday

1. Introduction.

The first half of this note concerns modules; so let R be a nonzero associative ring with unit. A countably infinite direct product of copies of R, which we will usually regard as a left R-module, will be written R^ω (ω denoting the set of natural numbers); the corresponding direct sum, i.e., the free left R-module of countably infinite rank, will be written $\bigoplus_\omega R$.

Here, now, are the two not-always-true statements of the title:

(1) There is no surjective left R-module homomorphism $\bigoplus_\omega R \to R^\omega$.

(2) There is no surjective left R-module homomorphism $R^\omega \to \bigoplus_\omega R$.

In §2 we will note classes of rings R for which each of these statements fails. In §§3-4, however, we will see that (1) holds, i.e., R^ω requires uncountably many generators as a left R-module, unless R^ω is *finitely* generated, and that (2) holds unless R has descending chain condition on finitely generated right ideals.

From the above assertion regarding (1), and the statement of (2), it is easy to see that for every R, at least one of (1), (2) holds. We shall also see that the above restriction on rings for which (2) fails implies that for every R, either (2) or the statement

(3) There is no embedding of left R-modules $R^\omega \to \bigoplus_\omega R$.

holds. (I did not count (3) among the "not quite true" statements of the title, because it does not appear that it is "nearly" true; i.e., that its failure implies strong restrictions on R.)

The result asserted above in connection with (1) will in fact be proved with R^ω replaced by M^ω for M any R-module, while the result on (2) will be obtained with R^ω generalized to the inverse limit of any countable inverse system of finitely generated R-modules and surjective homomorphisms.

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In §5 we change gears: We will note that our proof of the result about (1) generalizes to the context of general algebra (a.k.a. 'universal algebra'), and in §6, we deduce from this that if A is an algebra such that A^ω is countably generated over the diagonal image of A, then it is finitely generated over that image. We will then show that the monoid (respectively, the group) of all maps (respectively, invertible maps) of an infinite set Ω into itself has this finite generation property, and will obtain general results on algebras with this and related properties.

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2. Counterexamples.

It is easy to give rings for which statement (2) fails: If R is a division ring, then R^ω is infinite-dimensional as a left R-vector-space, hence admits a homomorphism onto $\bigoplus_\omega R$. More generally, if R is a quasi-Frobenius ring, then the submodule $\bigoplus_\omega R \subset R^\omega$, being free, is injective [**14**, first paragraph], so R^ω can be retracted onto it, and again (2) fails. The result to be proved in §4, that for (2) to fail R must have descending chain condition on finitely generated right ideals, shows that examples of failure of (2) are all fairly close to these.

Counterexamples to (1) are less evident. To construct these, we begin by noting that if M is a left module over any ring K and $(N_i)_{i\in I}$ any family of such modules, then under the standard convention (cf. [**1**]) that homomorphisms of left modules are written on the right of their arguments and composed accordingly, we have

(4) $\quad \mathrm{Hom}_K(\bigoplus_I N_i,\, M) \cong \prod_I \mathrm{Hom}_K(N_i,\, M)$ as *right* $\mathrm{End}_K(M)$-modules,

and

(5) $\quad \mathrm{Hom}_K(M,\, \prod_I N_i) \cong \prod_I \mathrm{Hom}_K(M,\, N_i)$ as *left* $\mathrm{End}_K(M)$-modules.

Indeed, the bijective correspondences follow from the universal properties of the direct sum in (4) and the direct product in (5); it remains only to note that M is a right $\mathrm{End}_K(M)$-module, and that this module structure carries over to the hom-sets of (4), while it is turned by contravariance into left module structures on the hom-sets of (5). We can now get our examples.

Lemma 1. *Let K be a ring and κ a cardinal. Then if M is either a* right *K-module which satisfies*

(6) $\quad M \cong \bigoplus_\kappa M$

(for instance, if κ is infinite and M a right K-module of the form $\bigoplus_\kappa N$), or a left *K-module which satisfies*

(7) $\quad M \cong M^\kappa$

(for instance, if κ is infinite and M a left K-module of the form N^κ), and if, in either case, we take $R = \mathrm{End}_K(M)$, then

(8) $\quad R \cong R^\kappa$ *as* left *R-modules.*

Proof. In (4) (with left and right interchanged), respectively (5) (as it stands), take $I = \kappa$, put the given module M in the role of both the M and all the N_i, and simplify the left hand side using (6), respectively (7). $\square$

A statement which embraces both cases of Lemma 1 is that if $\mathcal{C}$ is an $\mathcal{A}b$-category with an object M that is a κ-fold coproduct or product of itself, then its endomorphism ring in $\mathcal{C}$ (or the opposite of that ring, depending on which of these cases one is in, and one's choice of how morphisms are composed in $\mathcal{C}$) satisfies (8).

Clearly, when (8) holds, (1) and (3) fail.

Let me say here how this subject came up. H. W. Lenstra Jr., in connection with a course he was teaching in Leiden, e-mailed me asking whether, for a nontrivial ring R, one could have

(9) $\qquad R^\omega \;\cong\; \bigoplus_\omega R$

as left R-modules. I replied that this was impossible because R^ω could not be countably generated. He pointed out the first case of Lemma 1 (for M an abelian group, i.e., $K = \mathbb{Z}$), which shows the contrary. I then thought further and found the argument of the next section, showing that (9) nonetheless cannot occur.

I have not been able to find in the literature any occurrence of the idea of Lemma 1 for infinite κ, so it appears that the result belongs to Lenstra. However, two less elementary examples of similar phenomena were proven earlier. J. D. O'Neill [**25**] constructs for any $\kappa > 1$ a ring R without zero-divisors such that $R^\kappa \cong R^2$ as left modules, and as noted in the MR review of that paper, W. Stephenson showed in [**27**] that for any non-right-Ore ring S without zero divisors, the right quotient ring R of S satisfies (8) for any cardinal κ such that S has $\geq \kappa$ right linearly independent elements. That result is, in fact, an instance of the generalization noted immediately after the proof of Lemma 1 above. For the right quotient ring of S is the endomorphism ring of S in the $\mathcal{A}b$-category $\mathcal{C}$ whose objects are right S-modules, but where $\mathcal{C}(M,N)$ is the set of morphisms from essential submodules of M into N, modulo the equivalence relation that identifies morphisms which agree on essential submodules; and for a ring S without zero-divisors and having $\geq \kappa$ right linearly independent elements, the free right S-module on one generator has an essential submodule consisting of a direct sum of $\geq \kappa$ copies of S, leading to an isomorphism (6) in that category.

As suggested in the introduction, further counterexamples to (3) are not hard to come by. For instance, if, starting with any ring K, we form the noncommuting formal power series ring R in a κ-tuple of indeterminates, with no restriction on the number of monomials allowed in each degree, then its ideal of elements with constant term zero is isomorphic as a left module to R^κ, so R^κ embeds in R.

Since examples where (1) fails are harder to find, I will record here a slight extension of the class of examples given by Lemma 1.

First note that any ring R having a module isomorphism (8), e.g., a ring as in that lemma, will contain elements f_i $(i \in \kappa)$ such that the isomorphism is given by

(10) $\qquad r \;\mapsto\; (r f_i)_{i\in\kappa}.$

(For example, in a case arising from a right module isomorphism (6), $(f_i)_{i\in\kappa}$ can be any family of one-to-one endomorphisms of M such that $M = \bigoplus_\kappa f_i(M)$. In general, the f_i will be the components of the image under (8) of $1 \in R$.) But the surjectivity of (10) is preserved on replacing R by any homomorphic image; so such a homomorphic image will again be a ring for which (1) fails. To get rings R as in Lemma 1 which have ideals I such that R/I does not (so far as I know) itself satisfy (8), one can (a) let K be a field, R the endomorphism ring of an infinite-dimensional K-vector-space V, and I the ideal of finite-rank endomorphisms of V, or (b) let K be a ring having a non-finitely-generated 2-sided ideal I_0, let $R = \mathrm{End}_K(\bigoplus_\kappa K)$, and let I be the ideal of R generated by the diagonal image of I_0. Because I_0 is non-finitely-generated, I is generally smaller than the kernel of the natural map $\mathrm{End}_K(\bigoplus_\kappa K) \to \mathrm{End}_{K/I_0}(\bigoplus_\kappa K/I_0)$ (it does not contain endomorphisms whose components all lie in I_0, but do not all lie in any finitely generated subideal of I_0), so R/I is not simply the latter ring, which would just be another example of the construction of Lemma 1.

It is interesting to note that a ring R can satisfy (8) simultaneously as a right and as a left R-module (though necessarily by different bijections). Namely, if K is a division ring and κ an infinite cardinal, then the left vector-space $M = K^\kappa$ not only satisfies $M^\kappa \cong M$, but also, being a K-vector-space of dimension at least κ, satisfies $\bigoplus_\kappa M \cong M$; hence combining Lemma 1 and its dual, we get the desired right and left module isomorphisms.

We now turn to results showing that despite these instances of (8), no nontrivial ring satisfies (9).

3. A diagonal argument.

In this section, we shall restrict ourselves, for simplicity of presentation, to the case $\kappa = \omega$. (In §5, in addition to passing from module theory to general algebra, we will give the corresponding results for a general infinite cardinal κ.)

The hypothesis of the next lemma may seem irrelevant to the question at hand, but appearances are deceiving.

Lemma 2. *Let R be a ring and $(M_i)_{i\in\omega}$ a family of non-finitely-generated left R-modules. Then $\prod_{i\in\omega} M_i$ is not countably generated.*

Proof. It will suffice to show that for any countable family of elements $x_j \in \prod_{i\in\omega} M_i$ $(j \in \omega)$, we can construct an element y not in the submodule generated by the x_j. We do this by a diagonal construction: For each $i \in \omega$, the assumption that M_i is not finitely generated allows us to take as the i-component of y an element of M_i not in the span of the i-components of $x_0, \dots, x_i$. If y were in the span of all the x's, it would be in the span of finitely many of them, say $x_0, \dots, x_i$. But looking at its i-component, we get a contradiction. □

The relevance of that lemma is seen in the proof of

Theorem 3. *Let M be a left module over a ring R. Then the left R-module M^ω is either finitely generated, or not countably generated.*

Proof. If M^ω is not finitely generated, then the above lemma shows that $(M^\omega)^\omega$ is not countably generated. But $(M^\omega)^\omega \cong M^{\omega\times\omega} \cong M^\omega$, so M^ω is non-countably-generated, as claimed. □

So Lenstra's question is answered:

Corollary 4. *No nonzero ring R can satisfy $R^\omega \cong \bigoplus_\omega R$ as left R-modules.*

Proof. $\bigoplus_\omega R$ is neither finitely generated nor non-countably-generated, while we have seen that R^ω must have one of these properties. $\square$

In the context of Theorem 3, if M^ω is not finitely generated, it is natural to ask whether the least cardinality of a generating set must be $\geq 2^{\aleph_0}$.

Under some conditions, this is indeed true. Trivially, it is so if we assume the Continuum Hypothesis. Almost as trivially, writing $|X|$ for the cardinality of a set X, we see that if $|R| < |M|^{\aleph_0}$, in particular if $|R| < 2^{\aleph_0}$, then, even without an explicit assumption that M^ω is not finitely generated, as long as M is nonzero, a generating set for M^ω must have cardinality $|M|^{\aleph_0}$, since a module of infinite cardinality cannot be generated by a subset of smaller cardinality over a ring also having smaller cardinality.

Without any restriction on $|R|$, the module M^ω will *contain* a direct sum of $2^{\aleph_0}$ copies of M. This can be seen either by applying [**2**, Corollary 3.5] to the discrete topology on ω, or, alternatively, as follows: Note that $M^\omega \cong M^{\mathbb{Q}}$ where $\mathbb{Q}$ is the set of rational numbers. Associate to every real number r the map $f_r : M \to M^{\mathbb{Q}}$ carrying each $x \in M$ to the function on $\mathbb{Q}$ which has the value x at all rational numbers $q < r$ and 0 at all $q \geq r$, and verify that the sum of the resulting images of M is direct. For R a division ring, this shows that R^ω must have dimension at least $2^{\aleph_0}$; though in that case, the Erdős-Kaplansky Theorem [**16**, Theorem IX.2, p.246] gives the more precise result $\dim_R R^\omega = |R|^{\aleph_0}$.

However, Andreas Blass (personal communication) has pointed out to me how to get generation numbers strictly between $\aleph_0$ and $2^{\aleph_0}$. A modification of his argument gives the following result.

Proposition 5. *Let $\kappa < \lambda$ be infinite cardinals, with λ regular. Then there exists a ring R and an R-module M such that the R-module M^κ can be generated by λ but not by fewer elements. Moreover, R and M can be chosen either so that* (1) *M also requires λ generators, or, alternatively, so that* (2) *M is cyclic.*

Proof. Let K be any nonzero ring, and M the right K-module $\bigoplus_\lambda K$, whose canonical basis we shall write $(x_\alpha)_{\alpha\in\lambda}$. By the *support* of an element of M we shall mean the set of $\alpha \in \lambda$ such that x_α appears with nonzero K-coefficient in that element; the support of a subset of M will mean the union of the supports of its elements. Let R_1 be the subring of $\mathrm{End}_K(M)$ consisting of those endomorphisms that carry each x_α to an element with support in $\{\beta \mid \beta \leq \alpha\}$.

As a left R_1-module, M is generated by the λ elements x_α, but from the description of R_1 and the regularity of λ, we see that it cannot be generated by fewer. Hence M^κ, having M as a homomorphic image, also cannot be generated by fewer than λ elements.

On the other hand, for every $\alpha \in \lambda$, let $y_\alpha \in M^\kappa$ be the element whose β-coordinate, for each $\beta \in \kappa$, is $x_{\alpha+\beta}$ (sum as ordinals). I claim that $\{y_\alpha \mid \alpha \in \lambda\}$ generates M^κ. Indeed, any $z \in M^\kappa$ has only κ components, hence the support of the set of its components has cardinality $\leq \kappa$, and so is majorized by some $\alpha \in \lambda$. Then $z \in R_1 y_\alpha$; for the components of y_α are distinct elements $x_{\alpha+\beta}$, and each $\alpha + \beta$ majorizes the supports of all components of z, hence we may choose a member of R_1 to send each $x_{\alpha+\beta}$ to z_β, and thus send y_α to z.

To modify this example so that M becomes cyclic, let $I \subseteq \mathrm{End}_K(M)$ be the ideal of endomorphisms whose image has finite support, and let $R_2 = R_1 + I$. Clearly, $M = I x_0$, so a fortiori, $M = R_2 x_0$, showing M cyclic. M^κ is still generated over R_2 by the λ elements y_α, so it suffices to show that it is not generated by any subset $S_\gamma = \{y_\alpha \mid \alpha < \gamma\}$ with $\gamma \in \lambda$. And indeed, y_γ does not lie in the submodule generated by S_γ, for though the I-summands of finitely many elements of $R_2 = R_1 + I$ can boost components $x_{\alpha+\beta}$ of elements $y_\alpha \in S_\gamma$ up to the corresponding components $x_{\gamma+\beta}$ of y_γ for finitely many β, they cannot do so for infinitely many. $\square$

A restriction on this sort of behavior will be obtained in Proposition 26.

Returning to Theorem 3, let us note that the method by which we proved it from Lemma 2 can be applied to show uncountable generation of other sorts of product modules. For instance, if $(M_i)_{i\in\omega}$ is a family of finitely generated modules such that the finite numbers of generators they require is unbounded, it is clear that their direct product cannot be finitely generated. But such a family of modules can be partitioned into countably many infinite subfamilies each having the same unboundedness property; hence by Lemma 2, its direct product is in fact non-countably-generated.

This leaves open the case where the number of generators of the M_i is bounded. By passing to matrix rings, one can reduce this to the cyclic case, so we ask

Question 6. *Let R be a ring and $(M_i)_{i\in\omega}$ a family of cyclic left R-modules. Must the R-module $\prod_{i\in\omega} M_i$ either be finitely generated or require uncountably many generators?*

If $(M_i)_{i\in\omega}$ is any family of cyclic R-modules, let $V = \{S \subseteq \omega \mid \prod_{i\in S} M^i$ is not finitely generated$\}$. Whenever V contains a union $S \cup S'$ of two sets, it must clearly contain S or S', and if it contains infinitely many disjoint sets, an argument like that just noted shows that $\prod_{i\in\omega} M_i$ is uncountably generated. From this it is not hard to show that if $(M_i)_{i\in\omega}$ is a counterexample to Question 6, V must be a union of finitely many nonprincipal ultrafilters on ω. Taking $S \subseteq \omega$ which belongs to exactly one of these ultrafilters, and reindexing by ω, we get a counterexample where V itself is an ultrafilter. But I don't see how to go anywhere from there – the obvious thought is "look at the ultraproduct module $(\prod_{i\in\omega} M_i)/V$", but I see no reason why that module would have to be non-finitely-generated, let alone uncountably generated.

4. A touch of topology, and some chain conditions.

When I sent him Theorem 3, Lenstra noted that this implies not only that the left R-modules R^ω and $\bigoplus_\omega R$ are never isomorphic, but that one cannot map each surjectively to the other; i.e., in our present notation, that (1) and (2) cannot fail simultaneously. He then raised the question of whether for some R the module R^ω could admit both injective and surjective maps to $\bigoplus_\omega R$; i.e., whether (2) and (3) could both fail.

This is also impossible. I know now that this fact can be obtained without too much work from a result of S. U. Chase on homomorphisms from direct product modules to direct sums [**8**, Theorem 1.2]. However, I will prove some stronger statements, including a generalization of Chase's result, Theorem 14 below – essentially, the convex hull of his theorem and the similar result that I had obtained before learning of [**8**].

Before leaping into the proof, let us note that a general tool in proving restrictions on homomorphisms f out of an infinite product module $M = \prod_i M_i$ is to assume those restrictions fail, and construct an element $x \in M$ by specifying its values on successive coordinates, in such a way that the properties of $f(x)$ would lead to a contradiction. To "control" the effects of these successive specifications, one generally takes the ith coordinate to lie in an additive subgroup $I_i\, M_i \subseteq M_i$, using smaller and smaller right ideals $I_i \subseteq R$, and calling (explicitly or implicitly) on the fact that if a right ideal I is finitely generated, and we modify an arbitrary set of coordinates x_i of x by elements of $I\,M_i$, then this modifies $f(x)$ by an element of the subgroup $I\,f(M)$.

Both Chase's proof in [**8**] (and his proofs of similar results in [**6**] and [**7**]) and my original argument used this method of successive approximation. However, after a series of generalizations and reformulations, in which the property of the direct product being used was translated into a completeness condition with respect to an inverse limit topology, I realized that that step was essentially a repetition of the proof of the Baire Category Theorem, and could be avoided by calling on that theorem. Here is that part of the argument.

LEMMA 7 (cf. [**12**, Lemma 3.3.3]). *Let G be a complete metrizable topological group, and $(B_i)_{i\in\omega}$ a countable family of subgroups of G such that $\bigcup_{i\in\omega} B_i = G$. Then for some i, the closure of B_i in G is open.*

PROOF. The closed subgroups $\mathrm{cl}(B_i)$ again have union G, hence by the Baire Category Theorem, some $\mathrm{cl}(B_i)$ has nonempty interior, i.e., contains a neighborhood in G of one of its points x. By translation, it contains a neighborhood of each of its points, hence it is open. □

Although the preceding general observations concerned product modules, the same methods are applicable, more generally, to inverse limits of countable systems of modules, and we will prove our results for these. Note that if a module M is the inverse limit of a system

(11) $\quad \ldots \to M_i \to \ \ldots \ \to M_2 \to M_1 \to M_0$

with surjective connecting homomorphisms, then M maps surjectively to each M_i. (The corresponding statement is not true of inverse limits over uncountable partially ordered sets [**12**, Example 10.4]!) Hence each M_i may be written M/N_i, where the kernels form a chain

(12) $\quad N_0 \supseteq N_1 \supseteq \ldots \supseteq N_i \supseteq \ldots$, such that $\bigcap_{i\in\omega} N_i = \{0\}$, and M is complete in the (N_i)-adic topology.

(A countable *direct product* module $M = \prod_{i\in\omega} L_i$ is the case of (11) and (12) in which for all i, $M_i = \prod_{j=0}^{i-1} L_j$ and $N_i = \prod_{j\geq i} L_j$.)

The finitely generated right ideals I_i forming the other ingredient of the technique sketched above will come into the picture through the following curious result.

LEMMA 8 (cf. [**12**, Lemma 3.3.4]). *Let R be a ring, and M a left R-module having a chain of submodules* (12)*. Then for any chain of finitely generated right ideals of R,*

(13) $\quad I_0 \supseteq I_1 \supseteq \ldots \supseteq I_i \supseteq \ldots,$

M is also complete with respect to the chain of additive subgroups

(14) $\quad I_0N_0 \supseteq I_1N_1 \supseteq \ldots \supseteq I_iN_i \supseteq \ldots .$

PROOF. Let (x_i) be a sequence of elements of M satisfying

(15) $\quad x_{i+1} \in x_i + I_i\, N_i$ for all $i \in \omega$.

Since $I_i N_i \subseteq N_i$, the sequence (x_i) is Cauchy with respect to the (N_i)-adic topology, and so has a limit x in that topology. Our desired conclusion will follow if we can show that (x_i) converges to x in the $(I_i N_i)$-adic topology as well. To do this it will suffice to show that for each $i_0 \in \omega$,

(16) $\qquad x \ \in \ x_{i_0} + I_{i_0} N_{i_0}.$

Let us fix i_0 for the remainder of the proof, and establish (16).

We first note that (15) entails the weaker statement gotten by ignoring cases before the i_0th, and replacing all the ideals I_i $(i \geq i_0)$ with the larger ideal I_{i_0} :

(17) $\qquad x_{i+1} \ \in \ x_i + I_{i_0} N_i$ for $i \geq i_0$.

By assumption, I_{i_0} has a finite generating set S as a right ideal, so (17) shows that for each $i \geq i_0$ we can write

(18) $\qquad x_{i+1} \ = \ x_i + \sum_{s \in S} s\, y_{s,i}$, with $y_{s,i} \in N_i$ for each $s \in S$.

By completeness of M in the (N_i)-adic topology, for each $s \in S$ the series $\sum_{i \geq i_0} y_{s,i}$ converges in that topology to an element $y_s \in N_{i_0}$, so for each s, $\sum_{i \geq i_0} s\, y_{s,i}$ converges in the $(I_{i_0} N_i)$-adic topology to $s\, y_s \in I_{i_0} N_{i_0}$. Summing over S and adding to x_{i_0}, we conclude from (18) that the sequence (x_i), converges in that topology to

(19) $\qquad x_{i_0} + \sum_S s\, y_s.$

But by assumption, the sequence (x_i) converges in the (N_i)-adic topology to x, hence the limit (19) of that sequence in the stronger $(I_{i_0} N_i)$-adic must also be x, proving (16), as required. □

Now consider a situation where we have a homomorphism f from an inverse limit module M as above (for instance, R^ω) onto the free module of countable rank, $\bigoplus_\omega R$. For each $j \geq 0$, the elements $x \in M$ such that $f(x)$ has no nonzero components after the jth component form a submodule $B_j \subseteq M$, and since there are no elements x such that $f(x)$ has infinitely many nonzero components, $\bigcup_{j \in \omega} B_j = M$. In the next result, we use Lemma 7 to play the existence of such a chain of submodules off against completeness.

By a *downward directed system* $\mathcal{F}$ of right ideals of R, we shall mean a set $\mathcal{F}$ of right ideals such that every pair of members of $\mathcal{F}$ has a common lower bound in $\mathcal{F}$.

Theorem 9. *Let R be a ring and M a left R-module which has a chain of submodules* (12)*; equivalently, which is the inverse limit of a system* (11) *of modules and surjective homomorphisms. Suppose we are also given an ascending chain of submodules*

(20) $\qquad B_0 \ \subseteq \ B_1 \ \subseteq \ \dots$ *with union* M,

and a downward directed system $\mathcal{F}$ of right ideals of R. Then there exists $j^* \in \omega$ such that, writing q_{j^*} for the canonical map $M \to M/B_{j^*}$, we have

(21) $\qquad$ *The set* $\{I\, q_{j^*}(N_k) \mid I \in \mathcal{F},\ k \in \omega\}$ *of additive subgroups of* M/B_{j^*} *has a least member.*

I.e., there exists $I^* \in \mathcal{F}$ and $k^* \in \omega$ such that $I^*\, q_{j^*}(N_{k^*}) \subseteq I\, q_{j^*}(N_k)$ for all $I \in \mathcal{F},\ k \in \omega$.

Proof. Suppose (21) is false. We will begin by constructing recursively a sequence $I_0 \supseteq I_1 \supseteq \dots$ of ideals from $\mathcal{F}$, starting with an arbitrary $I_0 \in \mathcal{F}$, and a sequence $k_0 < k_1 < \dots$ of natural numbers, starting with $k_0 = 0$. Assuming

$I_0,\dots,I_{i-1}$ and $k_0,\dots,k_{i-1}$ constructed, the additive subgroup $I_{i-1}\,q_i(N_{k_{i-1}})\subseteq M/B_i$ is by assumption *not* least among subgroups $I\,q_i(N_k)$ $(I\in\mathcal{F},\ k\in\omega)$, so we can pick $I_i\subseteq I_{i-1}$ and $k_i>k_{i-1}$ giving a proper inclusion

(22) $$I_i\,q_i(N_{k_i})\ \subset\ I_{i-1}\,q_i(N_{k_{i-1}}).$$

Once these choices have been made for all i, Lemma 8 tells us that M is complete in the $(I_iN_{k_i})$-adic topology, hence by Lemma 7, there is some $j\in\omega$ such that the closure of B_j in that topology is also open. Thus for some $i_0\in\omega$, that closure contains $I_{i_0}N_{k_{i_0}}$, which is easily seen to imply that

(23) $$B_j+I_i\,N_{k_i}\ \supseteq\ I_{i_0}N_{k_{i_0}}\ \text{ for all } i\in\omega.$$

This means that when $i\geq i_0$, taking larger values of i (and hence smaller subgroups $I_i\,N_{k_i}$) does not decrease the left-hand side of (23). Also, note that (23), and hence that conclusion, is preserved on increasing j. We easily deduce

$$B_{j'}+I_i\,N_k\ =\ B_{j'}+I_{i'}\,N_{k'}\ \text{ for all } i,i'\geq i_0,\ j'\geq j,\ k,k'\geq k_{i_0}$$

or as an equation in $M/B_{j'}$,

$$I_i\,q_{j'}(N_k)\ =\ I_{i'}\,q_{j'}(N_{k'})\ \text{ for all } i,i'\geq i_0,\ j'\geq j,\ k,k'\geq k_{i_0}.$$

But the strict inclusion (22), for any $i\geq\max(j,i_0{+}1)$, contradicts this equality, completing the proof. □

The conclusion of the above theorem is rather complicated, with quantification over $\mathcal{F}$, $(B_j)_{j\in\omega}$ and $(N_k)_{k\in\omega}$. One can get a statement involving only the first two of these if one assumes that M is an inverse limit of *finitely generated* R-modules M_i.

COROLLARY 10. *Suppose in the situation of Theorem 9 that the R-modules $M_i=M/N_i$ of* (11) *are finitely generated. Then there exists $j^*\in\omega$ such that*

(24) *the set of additive subgroups $\{I\,(M/B_{j^*})\mid I\in\mathcal{F}\}$ of M/B_{j^*} has a least member.*

(*This will also be the least member of the larger family of additive subgroups described as in* (21).)

PROOF. Let j^*, I^*, k^* be as in Theorem 9, and note that the property asserted by that theorem is preserved under increasing j^* while leaving I^* and k^* unchanged. Let M/N_{k^*} be generated by the image of a finite set $T\subseteq M$. Since $\{B_j\}$ has union M, we can assume by increasing j^* if necessary that $T\subseteq B_{j^*}$. This says that the image of B_{j^*} in M/N_{k^*} is all of M/N_{k^*}, equivalently, that $M=B_{j^*}+N_{k^*}$, equivalently, that the image of N_{k^*} in M/B_{j^*} is all of M/B_{j^*}. Substituting M/B_{j*} for $q_{j*}(N_{k*})$ in the final assertion of Theorem 9 (with $k=k*$) gives the desired conclusion. □

We can now prove our earlier assertion about maps onto $\bigoplus_\omega R$.

COROLLARY 11. *Suppose R is a ring such that the free left R-module $\bigoplus_\omega R$ can be written as a homomorphic image of the inverse limit M of an inversely directed system* (11) *of finitely generated left R-modules and surjective homomorphisms. Then R is left perfect, i.e., has descending chain condition on finitely generated right ideals.*

In particular, any ring R for which there exists a surjective left R-module homomorphism $R^\omega\to\bigoplus_\omega R$, i.e., for which (2) *fails, is left perfect.*

Proof. Given M as above and a surjective homomorphism $f: M \to \bigoplus_\omega R$, let $B_i = f^{-1}(\bigoplus_0^{i-1} R)$ $(i \in \omega)$. Then $M = \bigcup B_i$; moreover, each factor module M/B_i is isomorphic to $\bigoplus_{j \geq i} R$, again a free left module of countable (hence nonzero) rank, so for every $i \in \omega$, distinct right ideals I yield distinct abelian groups $I(M/B_i)$. The conclusion of Corollary 10 therefore tells us that every downward directed system of finitely generated right ideals of R has a least member. Applied to chains of ideals, this says that R has DCC on finitely generated right ideals, as claimed. The final assertion is clear. □

We noted at the beginning of §2 that (2) always failed when R was a quasi-Frobenius ring; so the class of rings for which it fails lies between the classes of quasi-Frobenius and left perfect rings. It would be of interest to characterize this class more precisely.

H. Lenzing [**22**, Proposition 2, in left-right dual form] shows that if $\bigoplus_{i \in \omega} R \subseteq R^\omega$ is a direct summand as left modules, then R is semiprimary with ascending chain condition on left annihilator ideals. (Whether the converse holds is left open [**22**, sentence beginning at bottom of p.687].) The failure of our condition (2) is equivalent to the statement that $\bigoplus_{i \in \omega} R$ can be embedded in *some* way as a direct summand in R^ω; so it is natural to ask

Question 12. *If $\bigoplus_{i \in \omega} R$ has some embedding as a direct summand in R^ω, will the* canonical *copy of $\bigoplus_{i \in \omega} R$ in R^ω be a direct summand?*

Getting back to our original goal, we can now combine Corollary 11 with known results to obtain the result Lenstra asked about.

Corollary 13. *For a nonzero ring R, statements* (2) *and* (3) *cannot fail simultaneously; i.e., R^ω cannot be both mappable onto and embeddable in $\bigoplus_\omega R$.*

Proof. By the observations in the third paragraph following Corollary 4, a power module R^ω always contains a direct sum of $2^{\aleph_0}$ copies of R, hence if R^ω is embeddable in $\bigoplus_\omega R$, that module will contain a set X of $2^{\aleph_0}$ left linearly independent elements. Writing $\bigoplus_\omega R$ as the union of the countable ascending chain of submodules $\bigoplus_0^{i-1} R$, we conclude that for some i, the latter submodule will contain uncountably many members of X; so in particular it contains $i+1$ members of X; in other words, a free left R-module of rank i contains a free submodule of rank $i+1$. This leads to an $i{+}1 \times i$ matrix over R which is left regular (has zero left annihilator), hence to an $i{+}1 \times i{+}1$ matrix A with left linearly independent rows, but with a zero column. Thus, in the $i{+}1 \times i{+}1$ matrix ring over R, the element A is left but not right regular.

On the other hand, if R^ω is mappable onto $\bigoplus_\omega R$, then by the preceding corollary, R is left perfect, hence so is its $i{+}1 \times i{+}1$ matrix ring. A generalization by Lam [**20**, Exercise 21.23] of a result of Asano's says that the left regular elements, the right regular elements and the invertible elements in such a ring coincide, so an element A cannot be left but not right regular.

Hence the two conditions on R are incompatible. □

Returning to Theorem 9, let me now obtain from it a version with the conclusion in a form closer to Chase's formulation. Note that the hypothesis below is as in Theorem 9, except that instead of a countable ascending chain of submodules $(B_i)_{i \in \omega}$, we assume given a homomorphism from M into a module $\bigoplus_{\alpha \in J} C_\alpha$, with no restriction on the cardinality of the index set J.

THEOREM 14 (after Chase [**8**, Theorem 1.2]). *Let R be a ring and M a left R-module which has a descending chain of submodules* (12)*; equivalently, which is the inverse limit of a system* (11) *of modules and surjective homomorphisms, and let $\mathcal{F}$ be a downward directed system of right ideals of R. Suppose we are given a family $(C_\alpha)_{\alpha\in J}$ of left R-modules, and a homomorphism $f: M \to \bigoplus_{\alpha\in J} C_\alpha$. For each $\beta \in J$ let $\pi_\beta : \bigoplus_{\alpha\in J} C_\alpha \to C_\beta$ denote the βth projection map.*

Then there exist $k^* \in \omega$, a finite subset $J_0 \subseteq J$, and an $I^* \in \mathcal{F}$ such that

$$I^*\,\pi_\beta(f(N_{k^*})) \;\subseteq\; \textstyle\bigcap_{I\in\mathcal{F}} I\,C_\beta \text{ for all } \beta \in J - J_0. \tag{25}$$

PROOF. Assuming the contrary, let us construct recursively a chain of ideals $I_0 \supseteq I_1 \supseteq \dots$ in $\mathcal{F}$ and a sequence of indices $\alpha_0, \alpha_1, \dots \in J$, starting with arbitrary $I_0 \in \mathcal{F}$ and $\alpha_0 \in J$.

Say $I_0 \supseteq \dots \supseteq I_{i-1}$ and $\alpha_0, \dots, \alpha_{i-1}$ have been chosen for some $i > 0$. By assumption, the choices $I^* = I_{i-1},\ k^* = i,\ J_0 = \{\alpha_0, \dots, \alpha_{i-1}\}$ do not satisfy (25), hence we can choose some $\alpha_i \notin \{\alpha_0, \dots, \alpha_{i-1}\}$ such that $I_{i-1}\,\pi_{\alpha_i}(f(N_i)) \not\subseteq \bigcap_{I\in\mathcal{F}} I\,C_{\alpha_i}$. This in turn says that we can choose some $I_i \in \mathcal{F}$ such that

$$I_{i-1}\,\pi_{\alpha_i}(f(N_i)) \;\not\subseteq\; I_i\,C_{\alpha_i}. \tag{26}$$

Since this property is preserved under replacing I_i by a smaller ideal, and since $\mathcal{F}$ is downward directed, we may assume that $I_i \subseteq I_{i-1}$.

After constructing these ideals and indices for all i, we define for each $j \in \omega$

$$B_j \;=\; \{x \in M \mid \forall i \geq j,\ \pi_{\alpha_i}(f(x)) \in \textstyle\bigcap_{I\in\mathcal{F}} I\,C_{\alpha_i}\}.$$

Note that for any $x \in M$, the finite support of $f(x)$ contains α_i for only finitely many i, so taking $j \in \omega$ greater than these finitely many values, we see that $x \in B_j$. Thus, the ascending chain of submodules B_j has union M.

Applying Theorem 9, with $\{I_i \mid i \in \omega\}$ for $\mathcal{F}$, we get i^*, j^*, k^* such that in $q_{j^*}(M) = M/B_{j^*}$,

$$I_i(q_{j^*}(N_k)) \;=\; I_{i^*}(q_{j^*}(N_{k^*})) \text{ for all } i \geq i^*,\ k \geq k^*.$$

From the definition of B_{j^*} and q_{j^*}, this implies that

$$I_i\,\pi_{\alpha_j}(f(N_k)) + \textstyle\bigcap_{I\in\mathcal{F}} I\,C_{\alpha_j} \;=\; I_{i^*}\,\pi_{\alpha_j}(f(N_{k^*})) + \bigcap_{I\in\mathcal{F}} I\,C_{\alpha_j} \text{ for all } i \geq i^*,\ j \geq j^*,\ k \geq k^*.$$

Hence if we fix $j \geq j^*$ and $k \geq k^*$ and vary $i \geq i^*$, the left hand side above does not depend on the latter value. In particular, for $i = \max(i^*{+}1, j^*, k^*)$ we have

$$I_{i-1}\,\pi_{\alpha_i}(f(N_i)) + \textstyle\bigcap_{I\in\mathcal{F}} I\,C_{\alpha_i} \;=\; I_i\,\pi_{\alpha_i}(f(N_i)) + \bigcap_{I\in\mathcal{F}} I\,C_{\alpha_i},$$

so

$$I_{i-1}\,\pi_{\alpha_i}(f(N_i)) \subseteq\; I_i\,C_{\alpha_i} + \textstyle\bigcap_{I\in\mathcal{F}} I\,C_{\alpha_i} \;=\; I_i\,C_{\alpha_i},$$

contradicting (26) and completing the proof. □

As with Theorem 9, if one assumes the modules $M_i = M/N_i$ finitely generated, one gets a simplified conclusion, with $I^*\,\pi_\beta(f(M))$ in place of $I^*\,\pi_\beta(f(N_{k^*}))$.

Is my development above an improvement on Chase's proof of [**8**, Theorem 1.2]? It improves the result by generalizing direct products to inverse limits, and principal right ideals to finitely generated right ideals, but these changes could have been made without significantly altering his argument. The above development, with the auxiliary lemmas, and the derivation of Theorem 14 via Theorem 9, is longer than Chase's. The best I can say is that it provides alternative perspectives on what underlies these results, complementing those provided by the original proof.

For additional results of Chase and others on maps from direct product modules to direct sums, see [**6**], [**7**], [**24**], [**29**], and papers referred to in the latter two works. The more recent works obtain results on maps from products of not necessarily countable families of modules. It would be interesting to know whether similar results can be obtained for maps on inverse limits of general not necessarily countable inverse systems of modules.

Incidentally, reading Chase [**8**] led me to strengthen my own results by replacing an original descending chain of right ideals with a downward directed system $\mathcal{F}$, and to explicitly state Theorem 9 rather than passing directly to the case where the M_i are finitely generated, i.e., Corollary 10.

In [**29**], the right ideals occurring in these results are generalized still further, the operation of multiplying by such an ideal being replaced with any subfunctor of the forgetful functor from left R-modules to abelian groups that commutes with arbitrary direct products, and it is noted that these include not only functors of multiplication by finitely generated right ideals, but also functors obtainable from those by transfinite iteration.

Where above we have examined homomorphisms from direct products and related constructions to direct sums, [**12**] investigates homomorphisms from direct products and related constructions to an arbitrary fixed module, and [**13**] homomorphisms from an arbitrary fixed module to direct sums and related constructions; though in exchange for this greater generality, those papers study a more restricted set of questions. It is amusing that where, above, I took an argument by successive approximation and replaced it with an application of the Baire Category Theorem, the authors of [**12**] take a similar argument, presented in [**21**] in terms of the Baire Category Theorem, and translate it back into a construction by successive approximation. *Plus ça change . . . .*

Some similar results with nonabelian groups in place of modules are proved in [**15**], [**10**], [**11**].

5. Generating products of general algebras.

The diagonal argument by which we proved Theorem 3 used nothing specific to modules. Also, as noted at the beginning of §3, the focus there on countable products was for the sake of presentational simplicity. We shall give the result here in its natural generality. (However, for consistency of notation, having begun this paper in the context of module theory, I will not follow the general algebra convention of using different symbols for algebras and their underlying sets.)

As noted in the statement of the next theorem, though the hypothesis there restricts the *arities* of the operations, there is no restriction on the cardinality of the *set* of operations. That cardinality corresponds to the cardinality of the ring R in Theorem 3. The latter theorem is trivial for R of cardinality $< 2^{\aleph_0}$, and this one is likewise trivial if there are $< 2^{\kappa}$ operations.

The second conclusion of the theorem, concerning ultraproducts, was pointed out to me by T. Scanlon. (That conclusion formally subsumes the first conclusion, and of course implies the corresponding intermediate statement with U replaced by any filter containing no set of cardinality $< \kappa$, since any such filter extends to an ultrafilter with the same property.)

THEOREM 15. *Let κ be an infinite cardinal, and T a type of algebras such that all operations of T have arities $< \kappa$, and if κ is singular, all have arities $<$*

some common cardinal $<\kappa$; *but where no assumption is made on the cardinality of the set of those operations. Let* $(M_i)_{i\in\kappa}$ *be a* κ*-tuple of algebras of type* T, *each of which requires at least* κ *generators. Then* $\prod_{i\in\kappa} M_i$ *requires* $>\kappa$ *generators.*

In fact, if U *is any ultrafilter on* κ *which contains no set of cardinality* $<\kappa$, *then the ultraproduct* $(\prod_{i\in\kappa} M_i)/U$ *requires* $>\kappa$ *generators.*

Proof. The restriction on arities of operations of T insures that if an algebra of type T is generated by a set X, then each element of that algebra belongs to a subalgebra generated by $<\kappa$ elements of X, and hence that any family of $<\kappa$ elements is contained in such a subalgebra. Given this observation, the proof of the first assertion is exactly analogous to that of Lemma 2.

For $X \subseteq \prod M_i$ of cardinality $\leq \kappa$, y an element of $\prod M_i$ constructed, as in that proof, to avoid the subalgebra generated by X, and z any element of the latter subalgebra, note that y and z will in fact agree at fewer than κ coordinates. Hence for an ultrafilter U containing no set of cardinality $<\kappa$, the image of y in $(\prod M_i)/U$ will not equal the image of z; so $(\prod M_i)/U$ is not generated by the image of such a set X; i.e., it, too, requires $>\kappa$ generators. $\square$

We can now generalize Theorem 3. The value of λ of greatest interest in the next result is, of course, $\aleph_0$.

Theorem 16. *Let* $\lambda \leq \kappa$ *be infinite cardinals, with* λ *regular, let* T *be an algebra type such that all operations of* T *have arities* $<\lambda$, *and let* M *be any algebra of type* T. *Then the algebra* M^κ *either requires* $<\lambda$ *or* $>\kappa$ *generators.*

Proof. Let μ be the least cardinality of a generating set for M^κ. If μ were neither $<\lambda$ nor $>\kappa$, then noting that $M^\kappa \cong (M^\kappa)^\mu$, we could apply Theorem 15, with μ in place of κ and M^κ in place of each M_i, to conclude that M^κ requires $>\mu$ generators, a contradiction. $\square$

Unlike Theorem 15, Theorem 16 contains no statement about ultrapowers, so we ask

Question 17. *Is the analog of Theorem 16 true with* M^κ *replaced by the ultrapower* M^κ/U, *for* U *any ultrafilter on* κ *containing no set of cardinality* $<\kappa$*?*

If so, do all such ultrapowers of M *agree with one another as to whether the number of generators required is* $<\lambda$ *or* $>\kappa$*? If so, do they agree with* M^κ *in this respect?*

Something one *can* show is that if there is an ultrafilter U on κ such that M^κ/U can be generated by μ but no fewer generators, where $\lambda \leq \mu \leq \kappa$, then there is another ultrafilter U' such that M^κ/U' requires $>\mu$ generators. To see this, take any ultrafilter V on μ containing no subset with $<\mu$ elements. Then Theorem 15 applied with μ in the role of κ shows that $(M^\kappa/U)^\mu/V$ requires $>\mu$ generators. But there exists an ultrafilter UV on $\kappa\times\mu$ such that $(M^\kappa/U)^\mu/V \cong M^{\kappa\times\mu}/UV$; and since $\mu\leq\kappa$, we may choose an a bijection $\kappa\times\mu\to\kappa$, and this will take UV to the desired ultrafilter U' on κ.

I have no reason to expect a positive answer to the next question; the present wording is just the shortest way of asking for a counterexample that should exist, but that I don't know how to find.

QUESTION 18. *In the final conclusion of Theorem 16, can one strengthen "$<\lambda$" to "$<\aleph_0$"?*

Let us note a "difficulty" with Theorem 16 as a generalization of Theorem 3: Classes of algebras with infinitely many operations are not commonly considered in most fields other than ring- and module-theory. We will now note one close analog of the module-theoretic situation; then, in the next section, introduce a class of examples of a quite different sort, and obtain some results on these.

The nonlinear analog of an R-module is a G-set where G is a group or monoid, and the next lemma, which is proved just like Lemma 1, shows that in that context, the "$<\lambda$" case of Theorem 16 again occurs. In the category $\mathcal{C}$ referred to in the statement, morphisms are assumed to compose like functions written on the left of their arguments, i.e., the composite of morphisms $f : X \to Y$ and $g : Y \to Z$ is written $gf : X \to Z$.

LEMMA 19. *Let $\mathcal{C}$ be a category, κ a cardinal, and M an object of $\mathcal{C}$ which is either a κ-fold coproduct of copies of itself*

(27) $\quad M \cong \coprod_\kappa M$

(for instance, if κ is infinite and M any object of the form $\coprod_\kappa N$), or a κ-fold product of copies of itself

(28) $\quad M \cong \prod_\kappa M$

(for instance, if κ is infinite and M any object of the form $\prod_\kappa N$). In the former case, if we let G be the monoid $\mathrm{End}(M)$, or in the latter, if we let $G = \mathrm{End}(M)^{\mathrm{op}}$, then

(29) $\quad G \cong G^\kappa$ *as left G-sets.* $\square$

The situation is strikingly different when G is a group, no matter how constructed. If M is any left G-set of more than one element, then for every proper nonempty subset $S \subseteq \kappa$, the set A_S of elements of M^κ having exactly two values, one at all indices in S and the other at all indices in $\kappa - S$, is a union of G-orbits, and sets A_{S_1} and A_{S_2} are disjoint unless S_1 and S_2 are equal or complements of one another. So for κ infinite, M^κ consists of $\geq 2^\kappa$ orbits, and so cannot be generated by fewer elements.

The concept of a G-set, for G a monoid or a group, is generalized by that of an algebra M of any type on which G acts by endomorphisms, respectively by automorphisms. R-modules for R a ring are a particular class of examples; there may be others for which Theorem 16 would be of interest, and in particular, where the exotic phenomenon of generation of M^κ by $<\lambda$ elements occurs.

6. Generation of power algebras over their diagonal subalgebras.

A different way to get interesting examples of algebras with operation-sets of arbitrarily large cardinalities is to start with algebras of arbitrary type T, fix one such algebra X whose underlying set has large cardinality, and consider algebras Y given with a homomorphism i of X into them, formally treating the image of each element of X as a zeroary operation. The simplest case is that in which $Y = X$ and i is the identity map. For that case, Theorem 16 (with λ taken to be $\aleph_0$ for simplicity) says

Corollary 20. *Let κ be a cardinal, and M an algebra with operations all of finite arity. Then the algebra M^κ is either generated by the diagonal $\Delta(M)$ and finitely many additional elements, or requires $> \kappa$ additional elements.* $\square$

The next result gives, for any infinite cardinal κ, nontrivial algebras of two familiar sorts having only finitely many operations, whose κth direct powers are finitely generated over their diagonal subalgebras. In the proof, we continue to write maps on the left of their arguments and compose them accordingly, though this reverses the convention in the material from [**4**] cited in the proof of the second statement.

Theorem 21. *Let κ be an infinite cardinal and Ω a set of cardinality $\geq \kappa$.*
(i) *If M is the monoid of all maps $\Omega \to \Omega$, then M^κ can be generated over $\Delta(M)$ by two elements.*
(ii) *If S is the group of all permutations of Ω, then S^κ can be generated over $\Delta(S)$ by one element.*

Proof. (i): Since $|\Omega| = \kappa \cdot |\Omega|$, we can write Ω as the union of κ disjoint sets of the same cardinality as Ω, $\Omega = \bigcup_{i\in\kappa} \Sigma_i$. For each $i \in \kappa$, let $a_i \in M$ be a one-to-one map with image Σ_i, and $b_i \in M$ a left inverse to a_i. I claim M^κ is generated by $\Delta(M)$ and the two elements $a = (a_i)_{i\in\kappa}$ and $b = (b_i)_{i\in\kappa}$. For given any $x = (x_i)_{i\in\kappa} \in M^\kappa$, let us "encode" x in a single element $x' \in M$, defined to act on each subset Σ_i by $a_i\, x_i\, b_i$. Then we see that $b\,\Delta(x')\,a = (b_i\, x'\, a_i)_{i\in\omega} = (b_i\, a_i\, x_i\, b_i\, a_i)_{i\in\omega} = (x_i)_{i\in\omega} = x$, as required.

(ii): Again the idea will be to encode elements of S^κ in single elements of S. The trouble is that our structure no longer contains elements a_i and b_i giving bijections between Ω and the subsets of Ω on which we will encode the components of our κ-tuple. It will, however, contain permutations carrying each of these subsets bijectively to a common set $\Sigma_1 \subset \Omega$. Unfortunately, if we take an element whose actions on these subsets "encode" the coordinates of a member of S^κ, and conjugate it by each of these permutations to get permutations having those same actions on Σ_1, the information encoded in its other components will not disappear; it will move to other parts of Ω, where it will constitute "garbage" that we must get rid of. We will do this using commutator operations, in which those parts of our maps are commuted with identity maps. Finally, we will call on a result from [**4**] to go from permutations of Σ_1 to permutations of Ω.

So let $\Omega = \Sigma_1 \cup \Sigma_2$, where Σ_1 and Σ_2 are disjoint sets of the same cardinality as Ω, and let S_1 be the subgroup of S consisting of elements which fix all members of Σ_2 and act arbitrarily on Σ_1. We shall first show that S_1^κ is contained in the subgroup of S^κ generated over $\Delta(S)$ by a single element $f = (f_i)_{i\in\kappa}$.

Let us identify Σ_2 with $\Sigma_1 \times \kappa \times \{0,1\}$, as we may since $|\Omega| \geq \kappa$. For each $i \in \kappa$, we define an element $f_i \in S$ which cyclically permutes the three sets Σ_1, $\Sigma_1 \times \{i\} \times \{0\}$ and $\Sigma_1 \times \{i\} \times \{1\}$, and for each $j \in \kappa - \{i\}$ interchanges the two sets $\Sigma_1 \times \{j\} \times \{0\}$ and $\Sigma_1 \times \{j\} \times \{1\}$. Precisely, we let

$$f_i : \ \alpha \mapsto (\alpha, i, 0) \mapsto (\alpha, i, 1) \mapsto \alpha, \ \ (\alpha, j, 0) \leftrightarrow (\alpha, j, 1), \\ \text{for } \alpha \in \Sigma_1, \ j \neq i \text{ in } \kappa. \tag{30}$$

Now in the group of permutations of an infinite set, every element is a commutator by [**26**] or [**4**, Lemma 14], so given $x = (x_i)_{i\in\omega}$ in S_1^κ, let us, for each $i \in \kappa$, regard x_i as a permutation of Σ_1 and write it as the commutator of permutations u_i, v_i of that set. Let us now define elements $y, z \in S$ which act trivially on Σ_1

and on $\Sigma_1 \times \kappa \times \{1\}$, while on each set $\Sigma_1 \times \{i\} \times \{0\}$, y behaves like u_i and z like v_i, i.e., $y(\alpha, i, 0) = (u_i(\alpha), i, 0)$, $z(\alpha, i, 0) = (v_i(\alpha), i, 0)$. I claim that x is the commutator of $f^{-1}\Delta(y)f$ and $f^2\Delta(z)f^{-2}$. Indeed, looking at the ith component of $f^{-1}\Delta(y)f$ for any i, we see that it acts trivially except on Σ_1 and on the sets $\Sigma_1 \times \{j\} \times \{1\}$ with $j \neq i$, while $f^2\Delta(z)f^{-2}$ is trivial except on Σ_1 and on the sets $\Sigma_1 \times \{j\} \times \{0\}$ with $j \neq i$. Hence their commutator is trivial except on Σ_1, where it behaves as the commutator of u_i and v_i, i.e., as x_i. So the subgroup of S^κ generated by $\Delta(S) \cup \{f\}$ contains every $x \in S_1^\kappa$, as claimed.

For the remainder of the proof we put aside the description of Ω as $\Sigma_1 \cup (\Sigma_1 \times \kappa \times \{0, 1\})$ used to get this fact, and take a decomposition of a simpler sort, keeping Σ_1 as above, but choosing a set Σ_3 such that $\Sigma_1 \cup \Sigma_3 = \Omega$, and such that $\Sigma_1 \cap \Sigma_3$, $\Sigma_1 - \Sigma_3$ and $\Sigma_3 - \Sigma_1$ all have cardinality $|\Omega|$. From the proof of [**4**, Lemma 2] one finds that in this situation, if we denote by S_3 the group of permutations of Ω that fix all elements not in Σ_3, then every permutation of Ω can be written either as a member of the product-set $S_1\, S_3\, S_1$ or of the product-set $S_3\, S_1\, S_3$; hence we have $S = S_1\, S_3\, S_1\, S_3$. (See last three paragraphs proof of [**4**, Lemma 2]. If one doesn't want to read between the lines of that proof, one can use the statement of that lemma, which, for subsets U and V of S satisfying certain weaker conditions than being equal to the above subgroups, shows – after adjusting to left-action notation – that every element of S belongs either to $V(V\,U)^4$ or to $U(U\,V)^4$. Hence if U and V contain 1, every element will belong to $(U\,V)^6$, hence in our present situation, to $(S_1\, S_3)^6$, which one may use in place of $S_1\, S_3\, S_1\, S_3$ in the reasoning of the next sentence.) Now taking $t \in S$ which interchanges Σ_1 and Σ_3, we get $S_3 = t^{-1} S_1\, t$, so $S = S_1\,(t^{-1} S_1\, t)\, S_1\,(t^{-1} S_1\, t)$, hence $S^\kappa = S_1^\kappa\, \Delta(t^{-1}) S_1^\kappa\, \Delta(t)\, S_1^\kappa\, \Delta(t^{-1}) S_1^\kappa\, \Delta(t)$. Since the subgroup of S^κ generated by $\Delta(S) \cup \{f\}$ contains S_1^κ, the above equation shows that it is all of S^κ. □

The result from [**4**] called on in the last paragraph of the above proof was used there in proving two other properties of infinite symmetric groups, essentially (32) and (33) below. One may ask whether there is a direct relation between the conclusion of the above lemma and those properties. An implication in one direction is proved, under simplifying restrictions on the value of κ and the algebra-type, in the next result.

Theorem 22. *Let S be an algebra with finitely many primitive operations, all of finite arity, which satisfies*

(31) *The countable power algebra S^ω is finitely generated over the diagonal subalgebra $\Delta(S)$.*

Then S also satisfies both of

(32) *S cannot be written as the union of a countable chain of proper subalgebras.*

(33) *For every subset $X \subseteq S$ which generates S, there exists a positive integer n such that all elements of S can be represented by words of length $\leq n$ in the elements of X.*

Proof. It is not hard to show that the conjunction of (32) and (33) is equivalent to the statement

(34) Whenever $X_0 \subseteq X_1 \subseteq \dots$ is an ω-indexed chain of subsets of S with $\bigcup_{i\in\omega} X_i = S,$ such that for each primitive operation f of S and each $i \in \omega,$ one has $f(X_i, \dots, X_i) \subseteq X_{i+1},$ then $X_i = S$ for some $i \in \omega.$

So it suffices to show that (31) implies (34). (This reduction only needs the fact that (34) implies both (32) and (33). These implications are seen by taking for $X_i,$ in the first case, the ith member of an ω-indexed ascending chain of subalgebras, and in the second, the set of elements of S that can be represented by words of depth $\leq i$ in the elements of $X.$) We will prove this in contrapositive form, showing that if $(X_i)_{i\in\omega}$ is a family which satisfies the hypotheses of (34) but not the conclusion, and $Y \subseteq S^\omega$ is finite, then the subalgebra of S^ω generated by $\Delta(S) \cup Y$ must be proper.

Indeed, define the *rank* of an element $s \in S$ to be the least r such that $s \in X_r.$ Then our assumption that the conclusion of (34) fails says that the rank function is unbounded, so for each $i \in \omega$ let us choose an element x_i of rank at least $i + \max_{y\in Y} \operatorname{rank}(y_i)$ (where each $y \in Y \subseteq S^\omega$ is written $(y_i)_{i\in\omega}$). We claim that the element $x = (x_i)_{i\in\omega}$ does not lie in the subalgebra of S^ω generated by $\Delta(S) \cup Y.$

For if it did, it would lie in the subalgebra generated by $\Delta(Z) \cup Y$ for some finite subset $Z \subseteq S,$ and be represented by a word of some depth d in these elements; thus for each $i,$ x_i would be expressed as a word of depth d in the elements of Z and the ith components of the elements of $Y.$ We now get a contradiction on taking any $i > d + \max_{z\in Z} \operatorname{rank}(z),$ since then by choice of $x_i,$

$$\begin{aligned} \operatorname{rank}(x_i) \ &\geq\ i + \max\nolimits_{y\in Y} \operatorname{rank}(y_i) \\ &>\ d + \max\nolimits_{z\in Z} \operatorname{rank}(z) + \max\nolimits_{y\in Y} \operatorname{rank}(y_i), \end{aligned}$$

contradicting the existence of such a depth-d expression for $x_i.$ □

Conditions (32) and, more recently, (33) have been proved for several sorts of groups that arise in ways similar to infinite symmetric groups (see [**4**, §3] for references), and also for the endomorphism ring of a direct sum or product of infinitely many copies of any nontrivial module [**23**]. It seems likely that some or all of those proofs can be modified to establish (31) in these cases as well. The endomorphism-ring case, indeed, follows easily from Lemma 1 above and the observation that for any ring $R,$

(35) $$\begin{aligned} &R^\omega \text{ fin.gen. as left } R\text{-module} \implies R^\omega \text{ fin.gen. as } (R,R)\text{-bimodule} \\ &\qquad \implies R^\omega \text{ fin.gen. over } \Delta(R) \text{ as ring,} \end{aligned}$$

where the last step uses the fact that in $R^\omega,$ left or right module multiplication by an element of R is equivalent to ring multiplication by an element of $\Delta(R).$ In fact, since Lemma 1 shows that R^ω is cyclic as a left R-module, we have not only "finitely generated" but "generated by one element" in this case. Likewise, replacing rings and modules by monoids and M-sets, and the application of Lemma 1 with an application of Lemma 19, with $\mathcal{C} = \mathcal{S}et,$ we get Theorem 21(i) with "two elements" strengthened to "one element". (I stated Theorem 21(i) as I did because the proof given seemed the best way to lead up to the more difficult proof of part (ii). In proving directly the "one element" statement, one can take that one element to be what we called a in the earlier proof. Given $x = (x_i),$ if we define $x'' \in M$ to have restriction to each Σ_i given by $x_i b_i,$ we see that $\Delta(x'')a = x.$)

A nontrivial *finite* algebra clearly satisfies (32) and (33), but will not satisfy (31), since $\Delta(S)$ is finite while S^ω is uncountable; so the converse of Theorem 22 is false. On the other hand (still considering only algebras with finitely many operations, all of finite arities), we see that a finitely generated infinite algebra cannot satisfy (33), and a countably generated but not finitely generated algebra cannot satisfy (32), so any infinite algebra satisfying both (32) and (33) must be uncountable, raising the hope that all such algebras might satisfy (31). However, this is not so, for it is shown in [**9**] and in [**18**, Corollaire 18] (both generalizing a result in [**19**]) that an infinite direct power of a finite perfect group satisfies (32) and (33); but such a group cannot satisfy (31), since it admits a homomorphism onto a nontrivial finite group, and (31) clearly carries over to homomorphic images.

Since the proof of Theorem 22 constructs (when (32) or (33) fails) an element x which disagrees at all but finitely many coordinates with every element of the subalgebra generated by $\Delta(S)\cup Y,$ one can in fact say (using again the idea of the last sentence of Theorem 15) that for any nonprincipal ultrafilter $U,$ the image of such an element x in S^κ/U fails to lie in the image of the subalgebra generated by $\Delta(S)\cup Y.$ Thus, for U a nonprincipal ultrafilter on $\omega,$ one can insert in Theorem 22 the intermediate condition

(36) The ultrapower S^ω/U is finitely generated over the image of the diagonal subalgebra $\Delta(S),$

i.e., strengthen that theorem to say that $(31)\Rightarrow(36)\Rightarrow(32)\wedge(33).$

Unlike (31), but like (32) and (33), condition (36) is clearly satisfied by finite algebras. For groups, it is also preserved by the operation of pairwise direct product; this follows from the fact that if groups G_1 and G_2 are generated by subsets X_1 and $X_2,$ each containing 1, then $G_1\times G_2$ will be generated by $X_1\times X_2.$ Hence the direct product of the permutation group on an infinite set with any nontrivial finite group is an example of an infinite algebra that separates (31) from (36). The example mentioned earlier of an infinite direct power of a finite perfect group turns out to separate (36) from $(32)\wedge(33),$ in view of the next result.

Proposition 23. *If an algebra S satisfies* (36) *for some nonprincipal ultrafilter U on $\omega,$ then there is a* (*finite*) *upper bound on the cardinalities of finite homomorphic images of $S.$ Equivalently, S has a least congruence C such that S/C is finite.*

Proof. Let S be an algebra which does not have a least congruence C making S/C finite. We shall show that S does not satisfy (36).

Denote by $\mathcal{F}$ the class of all congruences C on S with finite quotient $S/C,$ and let $C_{\text{res.fin.}}=\bigcap_{C\in\mathcal{F}}C.$ Since (36) is preserved under passing to homomorphic images, we may divide out by $C_{\text{res.fin.}}$ and assume S residually finite, but still infinite.

I claim now that we can find a chain of congruences $C_0\supset C_1\supset\dots\in\mathcal{F}$ and a sequence of elements $u_0,u_1,\dots\in S$ such that $(u_i,u_{i+1})\in C_i-C_{i+1}$ for all $i.$ Indeed, let $C_0\in\mathcal{F}$ be arbitrary. Since S/C_0 is finite, some congruence class c_0 with respect to C_0 is infinite. Choose a congruence $C_1\subset C_0$ such that c_0 decomposes into more than one congruence class under $C_1.$ At least one of these must be infinite; choose such an infinite class $c_1\subset c_0,$ and repeat the process. For each $i,$ choose $u_i\in c_i-c_{i+1};$ then the relations $(u_i,u_{i+1})\in C_i-C_{i+1}$ clearly hold. Writing p_i for the canonical map $S \to S/C_i$, these relations say that for all i,

(37) $$p_{i+1}(u_i) \neq p_{i+1}(u_{i+1}) = p_{i+1}(u_{i+2}) = p_{i+1}(u_{i+3}) = \dots .$$

To show that (36) fails, let $Y \subseteq S^\omega$ be any finite subset; we must show that the image in S^ω/U of the subalgebra of S^ω generated by $\Delta(S) \cup Y$ is a proper subalgebra.

Since S/C_0 and Y are finite, and U is an ultrafilter, there exists $R_0 \in U$ such that for each $y = (y_j)_{j\in\omega} \in Y$, the ω-tuple $(p_0(y_j))_{j\in\omega} \in (S/C_0)^\omega$ becomes constant when j is restricted to $R_0 \subseteq \omega$. Likewise, we can find $R_1 \subseteq R_0$ in U such that for each $y \in Y$, $(p_1(y_j))_{j\in\omega}$ becomes constant when restricted to $j \in R_1$, and so on, getting a chain $R_0 \supseteq R_1 \supseteq \dots$ of members of U such that for each $i \in \omega$ and each $y \in Y$, the elements $p_i(y_j) \in S/C_i$ are the same for all $j \in R_i$. Also, since U is nonprincipal, we can along the way make sure, for each i, that $i \notin R_i$, so that $\bigcap_i R_i = \emptyset$.

Now for any i, j, and j' such that the j- and j'-components of all elements of Y have equal images under p_i, the same will be true for all elements of the subalgebra generated by $\Delta(S) \cup Y$; hence the above conditions on the R_i yield

(38) For all $z = (z_j)$ in the subalgebra of S^ω generated by $\Delta(S) \cup Y$, all $i \in \omega$, and all $j, j' \in R_i$, we have $p_i(z_j) = p_i(z_{j'})$.

We shall now construct an element $x \in S^\omega$ which does not agree modulo U with any element z satisfying (38), in other words, such that for any such z, and any $R \in U$,

(39) $$\{j \in \omega \mid x_j = z_j\} \not\supseteq R.$$

For each $j \in \omega$, let $i(j)$ be the greatest integer such that $j \in R_{i(j)}$ if $j \in R_0$, and -1 otherwise; and let x_j be $u_{i(j)-1}$ if $i(j) \geq 1$, arbitrary otherwise. To prove (39) given $R \in U$, take any $j \in R_1 \cap R$, and then any $j' \in R_{i(j)+1} \cap R$, noting that $1 \leq i(j) < i(j')$. By (38), with $i = i(j)$, we have $p_{i(j)}(z_j) = p_{i(j)}(z_{j'})$. On the other hand, x_j and $x_{j'}$ are $u_{i(j)-1}$ and $u_{i(j')-1}$ respectively, so (37) with $i = i(j) - 1$ shows that $p_{i(j)}(x_j) \neq p_{i(j)}(x_{j'})$. Thus, the jth and j'th components of x cannot *both* coincide with the corresponding components of z. Since j and j' both belong to R, this proves (39), as required. □

Since we do not know the answer to Question 17, even for $\lambda = \omega = \aleph_0$, we in particular do not know whether (36) is equivalent to

(40) The ultrapower S^ω/U is *countably* generated over the image of the diagonal subalgebra $\Delta(S)$,

so it is worth noting that in both the implication (36) $\Rightarrow$ (32) $\wedge$ (33) and the above proposition, the hypothesis (36) can be weakened to (40), with some extra work. (Key changes: If Y is countable, write it as the union of a chain $Y_0 \subseteq Y_1 \subseteq \dots$ of finite subsets. In the construction of the element x as in the proof of Theorem 22, for each i let x_i have rank $\geq i + \max_{y\in Y_i} \operatorname{rank}(y_i)$, using the finite set Y_i rather than all of Y. Likewise, in the proof of Proposition 23, construct the sets R_i so that the elements $p_i(y_j) \in S/C_i$ $(j \in R_i)$ are merely the same for each $y \in Y_i$.)

It is also natural to ask

QUESTION 24. *For a given algebra S, if* (36) *holds for some nonprincipal ultrafilter U, will it hold for all nonprincipal ultrafilters? If not, what implications can be obtained among these conditions for different U ?*

7. Still more conditions.

Khelif [**18**, Définition 5] introduces a property having an interesting similarity to (31). Given an algebra-type T and a natural number n, let $W(n,T)$ denote the set of all n-ary words in the operations of T. Note that if S is an algebra of type T, then an element of $W(n,T)^\omega$ induces an n-ary operation on S^ω. Khelif's definition (reformulated to bring out the parallelism with (31)) says that an algebra S "has property P*" if there exists a natural number n and an element $w=(w_i)_{i\in\omega}\in W(n,T)^\omega$ such that every element $x\in S^\omega$ is the value of w at some n-tuple of elements of $\Delta(S)$. In [**18**, Définition 4] he defines a weaker condition P, which says that there exists a natural number n, and a function η from the natural numbers to the natural numbers, such that for every $x\in S^\omega$ there exists $w\in W(n,T)^\omega$ whose ith component is a word of length $\leq \eta(i)$ for each i, such that, again, x is the value of w at an n-tuple of elements of $\Delta(S)$; and he shows that P (and hence also P*) implies $(32)\wedge(33)$.

Since there are only finitely many words of a given length, and there is a bound on the lengths of any finite set of words, Khelif's property P is equivalent to the statement that for some n one can associate to every natural number i a finite subset of $W_i\subseteq W(T,n)$ such that every element $x\in S^\omega$ is the value of some $w\in\prod_i W_i$ at some n-tuple of elements of $\Delta(S)$. This suggests a parallel generalization of (31), where the finite generating set is replaced by one that is merely "locally finite":

(41) There exists a sequence $(X_i)_{i\in\omega}$ of finite subsets $X_i\subseteq S$ such that S^ω is generated by $\Delta(S)\cup\prod_i X_i$.

And indeed, one sees that the proof of Theorem 22 (and its strengthening by the insertion of conditions (36) and (40)) works equally well under this weaker hypothesis, since the construction there of the sequence (x_i) called on the finiteness of the generating set Y only via the finiteness of the subsets $\{y_i \mid y\in Y\}$ $(i\in\omega)$.

One can also combine features of (31) or (41) and Khelif's conditions, allowing, for instance, finitely many elements of $W(T,n)^\omega$ and finitely many elements of S^ω to be used together in generating S^ω over $\Delta(S^\omega)$. This plethora of conditions leads one to wonder whether there is some small number of "core" conditions to which most or all of those we have mentioned are related in simple ways; e.g., by adding conditions such as "S has no finite homomorphic images", adding cases such as "all finite algebras", and/or varying some natural parameters in the conditions.

It is noted in [**4**, paragraph following Question 8] that no non-finitely-generated abelian group satisfies either (32) or (33), so Theorem 22 shows (in two ways) that no nontrivial abelian group satisfies (31). One may ask generally

Question 25. *What can be said about varieties of algebras containing non-finitely-generated algebras S that satisfy* (31)*?* (32)*?* (33)*?* (36)*?* (40)*? Khelif's P*? P? Further variants of these?*

Another condition, weaker than both (33) and Khelif's property P, and of a more elementary nature, which might nevertheless be of interest in the study of these conditions, is the statement that for every generating set X for S, there exist an integer n such that every element of S belongs to a subalgebra generated by $\leq n$ elements of X. This is satisfied by the abelian group Z_{p^∞}, with $n=1$ independent of X.

Turning in a different direction, observe that if an algebra S satisfies (31), then so will S^ω. For if we write $S^\omega = S'$, and identify $(S')^\omega$ with $S^{\omega\times\omega} \cong S^\omega$, then the diagonal subalgebra of $(S')^\omega$ contains the diagonal subalgebra of $S^{\omega\times\omega}$, over which (31) shows that it is finitely generated. In particular, S^ω satisfies (33); so given a finite set Y such that $\Delta(S)\cup Y$ generates S^ω as in (31), there will be a bound on the lengths of words needed to so express every element of S^ω. It would be interesting to know whether one can in general get all elements of S^ω using a single word, with specified positions in each of which a specified element of Y occurs, while arbitrary elements of $\Delta(S)$ are put in the other positions. The proof of Theorem 21 shows that this is so in the case of the monoid or group of all maps or invertible maps of an infinite set into itself.

To an algebra S satisfying (31), we can associate several natural-number-valued invariants witnessing that condition: The least cardinality of a set Y such that S^ω is generated by $\Delta(S)\cup Y$; the least n such that for some finite $Y\in S^\omega$ every element of S^ω can be expressed using words of length (or depth) $\leq n$ in terms of elements of $\Delta(S)\cup Y$, etc.. Note that if we have a sequence $S_0,\ S_1,\ \dots$ of algebras of the same type which each satisfy (31), but for which the sequence of values of such an invariant is unbounded, then the product algebra $\prod_{i\in\omega} S_i$ cannot satisfy (31), in contrast to the observation of the preceding paragraph on a direct power of one such S.

A confession about Theorem 22: The hypothesis that S had only finitely many primitive operations was not used in the proof. I made that assumption because it would be needed in places in our subsequent discussion, and because without it, the conclusion (33) of that theorem is *weaker* than optimal. The reader will not find it hard to verify that if S has *countably* many primitive operations, then for any generating set X, all elements of S may be obtained not merely using words of bounded length as in (33), but using such words in finitely many of these operations, and more generally, that if S has an arbitrarily large set of primitive operations, and this set is partitioned in any way into countably many subsets, then S can be obtained as in (33) using the operations in the union of finitely many of these subsets. (Key idea: In (34), replace the relation $f(X_i,\dots,X_i)\subseteq X_{i+1}$ by $f(X_i,\dots,X_i)\subseteq X_{i+a(f)}$, where a is a function from the set of primitive operations to the positive integers giving the partition into countably many subsets, and in place of "depth" in the proof, use a "weighted depth" that takes account of this function. Incidentally, this result is analogous to (32), since the latter can be translated as saying that whenever a generating set for S is partitioned into countably many subsets, the union of some finite number of these subsets is still a generating set.) For an example of an algebra S satisfying (31) with uncountably many primitive operations, such that no *finite* set of these suffices, let $S=\omega$, and let the set of primitive operations be ω^ω, i.e., the set of *all* unary operations on S. Then S^ω is generated under these operations by a single element, the identity function, but one cannot obtain all of S from the generating set $\{0\}$ using finitely many operations.

We saw in Theorem 15 that large direct products led to algebras requiring increased numbers of generators; but Proposition 5 showed that this increase could be arbitrarily small. The next result shows that when this growth is "too" small, this can have restrictive consequences.

Recall that $\mathfrak{d}$ denotes the cofinality of the partially ordered set ω^ω of all sequences of natural numbers under componentwise comparison, a cardinal somewhere between $\aleph_1$ and $2^{\aleph_0}$. (It called the "dominating number", and generally described slightly differently [**28**], but the definitions are equivalent [**5**, Remark 2.3].)

PROPOSITION 26. *Assume the cofinality* $\mathfrak{d}$ *of the partially ordered set* ω^ω *is a regular cardinal.*

Let T *be a type of finitary algebras* (*with no restriction on the cardinality of its set of operations*) *and* $(M_i)_{i\in I}$ *an infinite family of nonempty algebras of that type, such that* $\prod_{i\in I} M_i$ *can be generated by* $<\mathfrak{d}$ *elements* (*or more generally, such that* $\prod_{i\in I} M_i$ *cannot be written as the union of a well-ordered chain of* $\mathfrak{d}$ *proper subalgebras*). *Then all but finitely many of the* M_i *satisfy* (32) *and* (33).

PROOF. It follows from the assumption that $\mathfrak{d}$ is regular that if $\prod_{i\in I} M_i$ can be generated by $<\mathfrak{d}$ elements, then it cannot be written as the union of a well-ordered chain of $\mathfrak{d}$ proper subalgebras; so the latter conditions does indeed generalize the former. To complete the proof of the proposition, let us assume given a family of algebras $(M_i)_{i\in I}$ having an infinite subfamily, which we can clearly assume countable, $(M_{i(n)})_{n\in\omega}$, such that none of the $M_{i(n)}$ satisfy both (32) and (33), and show that $\prod_{i\in I} M_i$ *can* be written as such a union.

As noted in the proof of Theorem 22, the assumption that no $M_{i(n)}$ satisfies both (32) and (33) is equivalent to the statement that each $M_{i(n)}$ may be written $M_{i(n)} = \bigcup_{m\in\omega} X_{n,m}$ for some chain of proper subsets $\dots \subseteq X_{n,m} \subseteq X_{n,m+1} \subseteq \dots$, such that for each primitive operation f of T, and each $m\in\omega$, one has $f(X_{n,m},\dots,X_{n,m}) \subseteq X_{n,m+1}$. Choosing such a chain for each n, let us define for each $x\in\prod_{i\in I} M_i$ (where $x=(x_i)_I$) the element $\mathrm{rank}(x)\in\omega^\omega$ to have as nth component, for each $n\in\omega$, the least m such that $x_{i(n)}\in X_{n,m}$. Note that for f a primitive operation of T, say of arity r, and $x^{(1)},\dots,x^{(r)}\in\prod_{i\in I} M_i$, the condition just stated relating f and the sets $X_{m,n}$ shows that an upper bound for $\mathrm{rank}(f(x^{(1)},\dots,x^{(r)}))$ will be $(\mathrm{rank}(x^{(1)})\vee\dots\vee\mathrm{rank}(x^{(r)}))+\Delta(1)$, where $\vee$ denotes componentwise supremum, and $\Delta:\omega\to\omega^\omega$ is the diagonal map.

Now let $\{s_\alpha \mid \alpha\in\mathfrak{d}\}$ be a cofinal subset of ω^ω; and for each $\alpha\in\mathfrak{d}$, let $N_\alpha\subseteq\prod_{i\in I} M_i$ be the set of elements x such that $\mathrm{rank}(x)$ is majorized by some element $(s_{\beta_1}\vee\dots\vee s_{\beta_r})+\Delta(k)$ with $\beta_1,\dots,\beta_r\in\alpha$ and $k\in\omega$. From our bound on $\mathrm{rank}(f(x^{(1)},\dots,x^{(r)}))$, we see that each N_α is a subalgebra; and the map $\alpha\mapsto N_\alpha$ is clearly isotone. Since $\{s_\alpha \mid \alpha\in\mathfrak{d}\}$ is cofinal in ω^ω, the union of these subalgebras is all of $\prod_{i\in I} M_i$. Finally, given α, to see that the subalgebra N_α is proper, note that the cardinality of $\{s_\beta \mid \beta\in\alpha\}$ is $<\mathfrak{d}$, hence so is the cardinality of the set of elements $(s_{\beta_1}\vee\dots\vee s_{\beta_r})+\Delta(k)$ $(\beta_1,\dots,\beta_r\in\alpha,\ k\in\omega)$, hence that set is not cofinal in ω^ω, hence we can construct an $x\in\prod_{i\in I} M_i$ whose rank is not majorized by any member of that set, i.e., which is not in N_α. □

Some final miscellaneous remarks:

An easily described class of algebras that satisfy (33) but not (32) are the infinite algebras in which the value of each operation is always one of its arguments; for instance, an infinite chain regarded as a lattice, or an infinite set with no operations. These examples also show that if we generalize [**4**, Question 8], which asked whether a countably infinite group can satisfy (33), to algebras of arbitrary type, the answer is affirmative.

Although (31) and (32) are preserved under taking homomorphic images, and although the same is true of (33) in any algebra whose structure includes a structure of group, it is not true of (33) in general, for if S' is a homomorphic image of S, the inverse image of a generating set for S' may not be a generating set for S. For example, let S' be the semilattice of finite subsets of ω under union, and S the semilattice obtained by hanging from each $x \in S'$, a new element

(42) $\qquad x_0 < x ,$

with no new order-relations other than the consequences of (42), so that, in particular, the new elements are pairwise incomparable. The retraction $S \to S'$ taking each x_0 to x is easily seen to be a homomorphism, so S' is a homomorphic image of S. Now any generating set for S must contain the set of all join-irreducible elements, $\{\emptyset\} \cup \{x_0 \mid x \in S'\}$, and every element of S is a join of at most two members of this set, so S satisfies (33); but (33) fails for S', as shown by the generating set consisting of $\emptyset$ and all singletons.

Condition (33) has been called by some authors "the Bergman property", based on its introduction in [**4**]. Another possible name for an algebra with this property is an *impatient* algebra, since it says "If you're going to generate me, you have to do it in finitely many steps – I can't wait forever!"

Though (31) is, for simplicity's sake, formulated above only for the ω-fold direct power, the corresponding condition on a general power S^κ is, of course, of interest. (It implies the κ-analog of (32), called "cofinality $> \kappa$".)

And, of course, the diagonal subalgebra $\Delta(S)$ of a power S^κ was merely the most obvious example of the context introduced in the first paragraph of §6, which would be worth considering in greater generality.

Another kind of algebra structure involving an arbitrarily large set of operations, all of finite arity, is a set on which a Boolean ring B "acts", in the sense of [**3**]. So Theorem 16 also applies to these structures, and it might be of interest to examine that case, possibly combining the Boolean operations with others.

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