BIOGRAPHICAL MEMOIRS OF FELLOWS OF THE ROYAL SOCIETY

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George Bergman and Trevor Stuart

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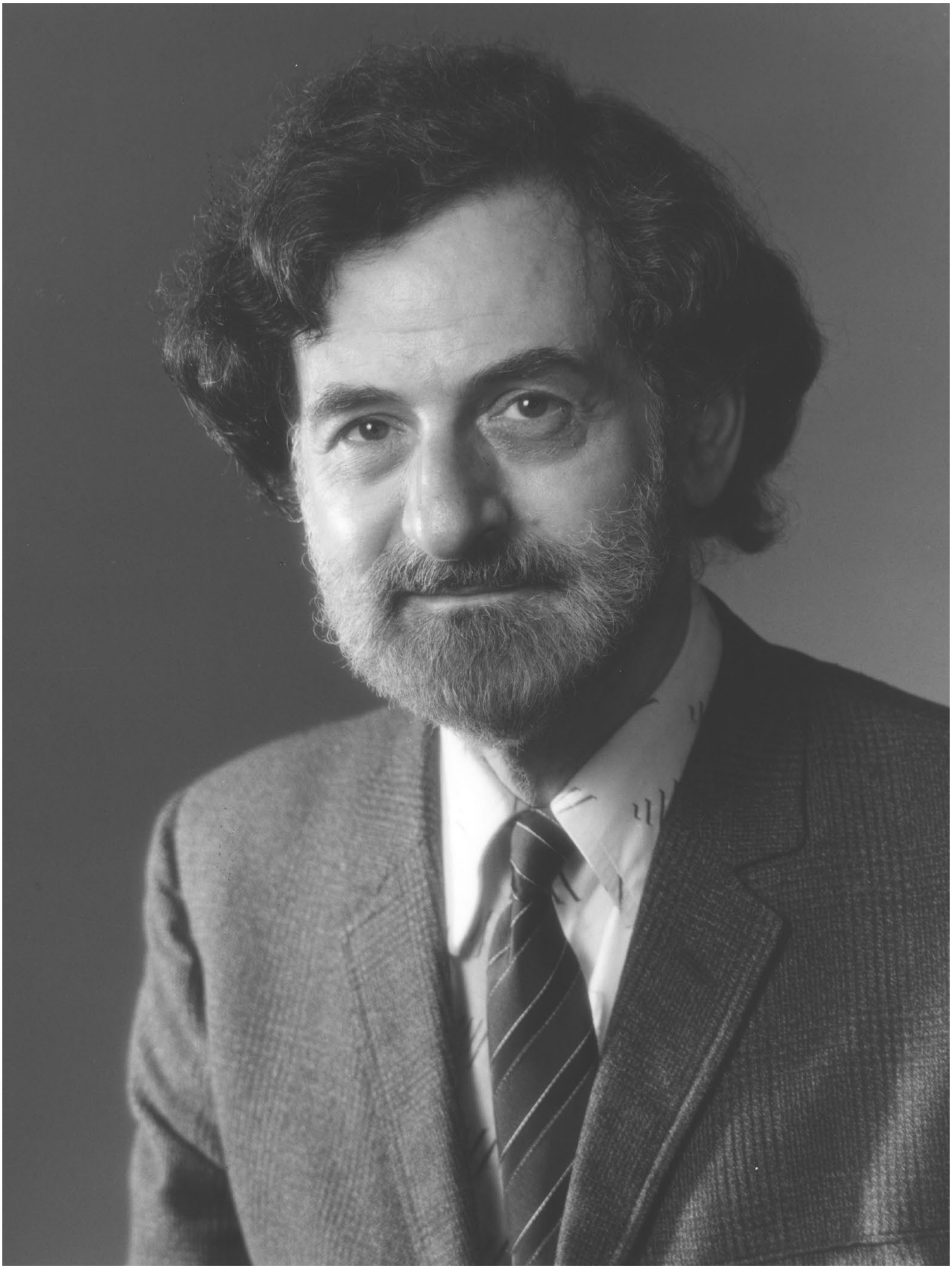

Paul Cohen

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Elected FRS 1980

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Paul Cohn was born in Hamburg, where he lived until he was 15 years of age. However, in 1939, after the rise of the Nazis and the growing persecution of the Jews, his parents, James and Julia Cohn, sent him to England by Kindertransport. They remained behind and Paul never saw them again; they perished in concentration camps. In England, being only 15 years old, he was directed to work first on a chicken farm but later as a fitter in a London factory. His academic talents became clear and he was encouraged by the refugee committee in Dorking and by others to continue his education by studying for the English School Certificate Examinations to sit the Cambridge Entrance Examination. He was awarded an Exhibition to study mathematics at Trinity College. After receiving his PhD in 1951, Paul Cohn went from strength to strength in algebra and not only became a world leader in non-commutative ring theory but also made important contributions to group theory, Lie rings and semigroups. He was much admired, and he travelled widely to collaborate with other algebraists. Moreover, he was a great supporter of the London Mathematical Society, serving as its President from 1982 to 1984.

PART A. PERSONAL AND FAMILY, BY TREVOR STUART

1. INTRODUCTION

Paul Cohn wrote at least two interesting accounts of his early life. One (28)*, which focused on his years in Germany, was published in German but an English translation is available as electronic supplementary material at http://rsbm.royalsocietypublishing.org/content/suppl/2014/08/14/rsbm.2014.0016.DC1/rsbm20140016supp2.pdf. The other (22), which is concerned with his early years in England, is in English.

* Numbers in this form refer to the bibliography at the end of the text.

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2. Early life and career

It may be helpful to start by summarizing parts of Paul's own accounts. Moreover, Juliet (Aaronson) Cohn, after discussions with her mother, Deirdre, and sister, Yael, has written some additional details of her father's life and these also are used throughout this section but without precise quotations.

Paul Cohn's parents were born in Hamburg, as were three of his grandparents. Earlier generations came from Hamburg, Leipzig, Berlin and Greiffenberg but, so far as he was able to trace, always from Germany. They considered themselves German (at least until 1933). His father was Jacob Cohn (1883–1942), but he was always known as James, and his mother was Julia Mathilde Cohn, *née* Cohen (1888–1941). His father joined the cigar-importing business belonging to his father-in-law in 1921. His mother had been a teacher since the age of 20 years. Paul was born on 8 January 1924 in Hamburg and was an only child. Initially his parents lived with his maternal grandmother but, when she died in 1925, they moved to a rented flat in a new building in the district of Winterhude. The front of the building overlooked the elevated railway, which Paul frequently observed, but the rear of their home overlooked a laundry yard with a small adjacent chicken run. The cock crowed in the morning like an illustration in Paul's favourite book, *Max and Moritz*, by Wilhelm Busch. In 1928 Paul developed scarlet fever and was taken to hospital by horse and carriage. (Scarlet fever was a serious complaint for children in the 1920s and 1930s in the UK as well as Germany; I had a similar experience in England.) In April 1930 Paul entered school (Alsterdorfer Strasse School); he enjoyed the lessons but was often teased in breaks, although anti-Semitism seemed to play no part. Later he was in a class with a teacher who continually picked on him and punished him without cause. His parents learned from the school head that his teacher was a National Socialist; they moved him to Meerweinstrasse School, which was where his mother taught. It was progressive and coeducational. Paul recalled that in 1932 there was a small sensation when one or two boys came to school in Nazi uniforms, but without repercussions.

His father's cigar-importing business declined in the years during Germany's Depression in the 1930s and was wound up in 1933, as people were discouraged from trading with Jews. Thus Paul's parents intended him to train as an optometrist or something similar. In 1933 Paul's mother was dismissed from her teaching position after the introduction of new legislation that removed Jews from the civil service. Paul's parents decided to send him to a Jewish school in Hamburg in the Grindel quarter, the Talmud-Tora-Schule. However, his mother was advised that he needed to improve his knowledge of mathematics so he worked flat out (just as he did much later in studying in England). The German lessons by Dr Ernst Loewenberg, the son of the poet Jacob Loewenberg, gave Paul a knowledge of and predilection for his native language that he never lost.

After Kristallnacht in 1938 his father was arrested along with many other Jewish men in Germany. Later his war record enabled his mother to have him released from the Sachsenhausen concentration camp; even so his parents were deported in 1941 to a concentration camp in Riga. (Paul's father had been awarded the Iron Cross in World War I after he rescued wounded comrades while under fire. A consequence was that Paul played without toy guns or other military toys as his father's experiences led to a hatred of war and all things associated with it.) After Julia's unsuccessful attempts to find any country to which they could all migrate, James and Julia Cohn found a place in 1939 for Paul on a Kindertransport to England, as he satisfied the requirement of being under 17 years old. However, they themselves had to remain in Germany and never saw their son again, perishing in concentration camps.

On arrival in England in May 1939 by Kindertransport, Paul was greeted at Liverpool Street Station in London by Mrs Lisbet Mueller-Hartmann, whom he remembered well as a distant relation from Hamburg. From there she escorted him by Underground to Victoria Station and arranged for him to take the train to Dorking, Surrey; there he was met by a lady from the refugee committee, who drove him to a farm at Newdigate, where Mr and Mrs Panning kept about 5000 chickens. Being over the school age of 14 years, Paul was required to work on the farm (unpaid, as this was a requirement of being accepted for Kindertransport). As he later recalled, before he left Germany he had told his father that he was fine with that, so long as he did not have to kill any chickens. Needless to say within a week he was doing so. This was a sort of introduction to British ways! His cousin, Peter, who worked on another farm, likewise unpaid, told him of being given a small amount of pocket money each week. This would probably have been one or two shillings and some pence (with 12 pence in one shilling and 20 shillings in £1), something else of British ways that Paul would have learned. Therefore Paul asked Mr and Mrs Panning for a similar amount, which was granted at the rate of 2 shillings and 6 pence (known as half a crown; in decimal currency 12½ pence) per week, this amount being increased gradually. Paul worked for 70 hours per week, with three half days off every two weeks. He would spend his pocket money at the local cinema, for which the charge was 6 pence, watching the same film many times to improve his use of English. Paul corresponded regularly with his parents during the summer of 1939, pursuing the possibility of their being accepted as immigrants to the UK if they gained employment as housekeeper and gardener (his father was an enthusiastic allotment gardener in Hamburg). His efforts were unsuccessful, and when war broke out all possibility of emigration ended. From this point onwards he received only a short letter once a month from his parents through the Red Cross. The letters became less frequent and stopped in late 1941. In fact they were deported to concentration camps in Riga on 6 December 1941.

However, the government intended that Paul should work for the duration of the war, but would then send him to Canada or Australia when the Atlantic Ocean was free of the dangers of U-boats. This transfer never happened and he stayed in England. It is an interesting reflection that had his future been determined otherwise, we might have been celebrating a Canadian or Australian mathematician!

It seems that because of his being on the young side (less than 16 years old), Paul was not sent as an alien to the Isle of Man, as were three musicians who later formed three-quarters of the Amadeus String Quartet! As the result of a shortage of feedstuff the small farm was eventually not viable, and Paul was saddened by the auction that followed closure. However, he was given a work permit and moved to London, where he worked as a fitter in a factory. He was still in touch with the refugee committee in Dorking, which, recognizing his intelligence and love of learning, and his special interest in mathematics, encouraged him to study for the Cambridge Entrance Examination and the School Certificate and Higher School Certificate Examinations, all of which he did by correspondence course. At that time Latin was still a requirement to enter the University of Cambridge, so he studied Latin from scratch. During his studies in his unheated room and before he started work, Paul needed to heat up his pen because the ink in it would freeze during the winter. (I am reminded of similar conditions in the 1940s due to coal shortages.) Thus his experience was not atypical.

Paul gained an Exhibition to study mathematics at Trinity College. He was released from the factory in 1944 and went up to Cambridge, but was asked to return to the factory after one

Figure 1. Paul Cohn in Cambridge in about 1950. (Copyright © Ramsey & Muspratt; reproduced with permission.)

term because he had been released in error. In spite of having had only one term of full-time study he passed the first-year examinations. He was finally released from the factory after one more year and resumed his studies in Cambridge.

Paul graduated with a BA in 1948 and then undertook research in algebra with financial support from a Department of Scientific and Industrial Research award. He was supervised by Philip Hall FRS and studied rings and free groups, obtaining his PhD in 1951 (figure 1). For the following academic year he was a Chargé de Recherche at the Université de Nancy. He became a lecturer at the University of Manchester in 1952. In 1962 he joined the University of London, initially at Queen Mary College until 1967, but after a year's sabbatical in Princeton, NJ, he moved in 1968 to Bedford College in its idyllic setting in Regent's Park. In 1984 he transferred to University College London, together with several other staff from Bedford College. He succeeded Professor Ambrose Rogers FRS as Astor Professor of Mathematics in 1986 and became Emeritus Professor in 1989. During his career in the field of algebra, in which he became a popular and influential figure, he visited many universities abroad, including Chicago (6 months in 1964), Novosibirsk, Rutgers, Princeton, Berkeley, Paris, Tulane, the Indian Institute of Technology in Delhi, Alberta, Carleton in Ottawa, the Technion, Iowa State University, Bielefeld in the country of his birth, and Bar Ilan University.

Paul's time in Manchester had two distinct benefits (in no particular order): (i) his research work and his reputation as an algebraist developed greatly; (ii) he had the pleasure of meeting Deirdre Sharon, who was a psychology undergraduate and who was to become his wife. They were married in 1958 and had two daughters, Juliet and Yael, each of whom read mathematics at university, Juliet at Trinity College, Cambridge, and Yael at Somerville College, Oxford

(Mary Somerville, after whom the college was named, was a mathematician in the nineteenth century). Paul had five grandchildren: James, Olivia and Hugo Aaronson, and Malke and Rusi Rappaport. It would have given him great pleasure had he known the interest his grandchildren take in mathematics. The eldest, James, following in his grandfather's footsteps, is currently studying mathematics at Trinity College, University of Cambridge.

Paul was fascinated by the study of language and its origins and was enthusiastic to learn something of the languages of the countries he visited; for example, he learned some Hindi when flying to Delhi. As to reading he enjoyed German philosophers and German literature, and Mozart was a favourite composer of his. He enjoyed also watching the Marx Brothers. In spite of having few relations to consult, Paul Cohn was very interested in his family ancestry and visited institutions of record—libraries, cemeteries, tombstones, and not least in Hamburg—so as to learn more of his heritage. He found that over the years many ancestors had a love of learning and scholarship; his genealogical research enabled him to trace his ancestors back to 1450.

Science in the UK benefited enormously from refugees such as Paul Cohn, who came to our shores as a safe haven, for which Paul was always grateful. In addition he felt gratitude to the refugee committee in Dorking, and to the composer Ralph Vaughan Williams, for encouraging and helping him to find his passion in mathematics. It was Vaughan Williams who particularly encouraged him to study mathematics at Trinity College in Cambridge. This passion for mathematics took him to many far-flung places, where people would have a different way of looking at a particular mathematical problem. He was always very excited to meet and talk to others who shared his passion for algebra, and indeed any branch of mathematics.

Paul did not feel completely English, but was always grateful to the country that had saved his life and given him a home in England. Another refugee, Harry Reuter, had views, as expressed in conversation to me, which in some ways were not unlike those of Paul Cohn. (For Harry Reuter's background, see https://de.wikipedia.org/wiki/Harry_Reuter.)

3. Research students, colleagues and research visits

According to the Mathematics Genealogy Project (http://www.genealogy.math.ndsu.nodak.edu/id.php?id=27131), Paul Cohn had 17 students and at the time of writing has more than 40 mathematical descendants with the number growing (students, students of students, and so on). His students were wholly within the University of London. Although Paul Cohn was not the official supervisor of George Bergman, the co-author of this memoir, he was a member of his dissertation committee and provided enormous help. For example, Bergman would mail him the draft of his thesis in instalments of about 25 pages and Paul would typically send a five-page letter of suggestions and corrections. In addition, Cohn arranged for him to visit Bedford College in 1969–70, where Bergman had a very productive year. They continued to correspond well into the twenty-first century.

We describe below some recollections received from Paul's students. Aidan Schofield was the author of an obituary notice in *The Independent* of 8 August 2006 in which he wrote:

> He was devoted to research which was profound, original and lonely, choosing to work on what he felt to be important rather than to follow fashionable trends in mathematics. In the early sixties he set himself a particular task and, 20 years later, he had completed it. In doing so he uncovered

> mathematical structures whose importance is beginning to be recognized in areas apparently far removed from the algebras and skew fields with which he worked.

On a more personal note, Schofield wrote:

> In 1980 I became Paul Cohn's graduate student at Bedford College in Regent's Park. His books had been an inspiration and delight for me and he himself fulfilled all I had been led to expect from them. His books reflected a scholarly and gentle approach. He was always willing to take time to talk about mathematics and he had a desire to explain what he had seen to those that had not. He also wanted to hear what others had to say and took pleasure in the successes of those around him.

Another of his students, Muhammad Zafrullah, has written to us with interesting anecdotes about Paul, of whom he writes affectionately as 'my Mathematical Old Man'! He started working with Paul in 1971. He comments on the important contribution that Paul made to mathematics through education and training: 'He was very strict over those he accepted as students and then trained them to think and work independently, giving pointers from time to time on how to proceed in certain situations.' At one point in his research, Muhammad sensed that Paul was uncomfortable with his working on commutative topics independently, with Paul working on non-commutative algebra. Paul Cohn therefore arranged for Masayoshi Nagata to visit the college. 'The meeting with Nagata did wonders for me', writes Zafrullah, saying further, 'Paul did so much to help his students.' When Paul died, Muhammad felt his loss greatly.

A student from Paul's early days at Queen Mary College was Bill Stephenson, who comments, 'Paul was an assiduous/meticulous supervisor.' Sometimes during his regular supervision sessions with Paul, Bill would raise topics in Algebra that were not necessarily relevant for his thesis, but 'Paul could straightaway go into this area I had asked about. He really was Mr Algebra.' After gaining his PhD in 1966, Bill spent a year in Russia before joining the staff in 1967 at Bedford College, to which Paul had moved. However, Bill became involved in trade unions in the late 1960s, during which time Paul teased him with good humour about this interest and commented that he himself had actually been a union member in the 1940s when he worked in a factory. Bill also writes that he was 'genuinely pleased when he heard that Paul had become an FRS.' He finishes by saying, 'I owe so much to Paul for supervising me, appointing me and standing by me in difficult times.'

Some recollections from other university colleagues and from research associates in the USA and Russia are now described. Another member of staff at Bedford College was Wilfrid Hodges, who was Reader in Mathematical Logic until in 1987 he moved with the same title to Queen Mary College, becoming later Professor of Mathematics and Dean of Mathematical Sciences there. He writes:

> When I first came to Bedford College in 1968, it was to join the Philosophy Department in the first instance. But the Head of Philosophy, David Wiggins, knew of my interest in mathematics and approached Paul Cohn about the possibility of my being a joint lecturer in Philosophy and Mathematics. Paul took a big gamble on this since, at that stage, I had no mathematical qualifications beyond A level. But it worked out, and in 1974 I joined the Mathematics Department full time. I am hugely in debt to Paul for allowing all this.

He also says, 'Paul always treated me as a fellow researcher like himself' and 'from early days he quoted remarks of mine in his papers and books, which certainly helped to get my name known in the algebraic community. He also strongly encouraged me to set up a model-theory

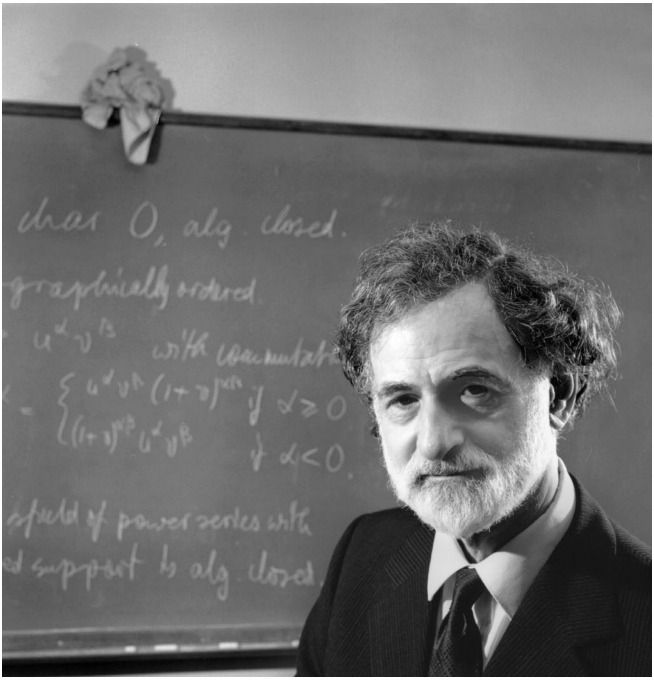

Figure 2. Paul Cohn at a blackboard in 1989.

research group.' At one of the regular meetings for morning coffee, Wilfrid Hodges recollects 'Paul announcing that his books had sold 100,000 copies!' With reference to the move from Bedford College upon its closure and merger with Royal Holloway College, he says 'Paul took his Bedford team to University College London to form part of the algebra unit there.' Wilfrid Hodges in logic moved to Queen Mary College. George Bergman recalls that at the same time Paul arranged for the Departmental Secretary, Eileen Simpson, to move to a staff position at the London Mathematical Society (LMS).

In relation to Bedford College and his colleagues (25), Cohn writes:

> Reflecting on my seventeen years with the College, I found the atmosphere, both physical, in one of London's finest parks, and mental, surrounded by colleagues who were both stimulating and sympathetic, very conducive to productive research and rewarding teaching.

A photograph of Paul Cohn at the blackboard is shown in figure 2.

I have mentioned earlier the pleasure that Paul Cohn gained from visits to other universities and from colleagues whom he met and with whom he discussed mathematics, and whose friendship he gained. One such colleague was George Bergman; another was Professor Carl Faith, who has commented on the pleasure that Paul gained from the copious open spaces of Princeton and his remark that they seemed 'wasted, that is, uncultivated.' 'The open spaces are still there', writes Carl, with the 'Institute for Advanced Study holding over 100 acres of woods for members to trample through', together with the Veblen Arboretum and another patch of woodland under the aegis of the Audubon Society. Princeton University and the Institute are contiguous with Lake Carnegie. Carl Faith writes that when 'Paul arrived in Princeton with his wife, Deirdre, his two daughters and two students, he enriched our lives immeasurably.'

Carl Faith wrote a 'long appreciative review in *Mathematical Reviews* about Cohn's solution of Artin's problem'. Paul told Carl of the happiness that this gave him. Faith writes, 'Many had previously tried to solve Artin's problem but no-one had done so before!' See Cohn's paper (4) and the review by Faith (1963).

Paul Cohn had a long association with Leonid A. Bokut, who writes (in a letter translated by George Bergman), 'at Moscow State University my academic advisor A. I. Shirshov drew my attention to the work of Cohn', which, 'together with the work of Shirshov, became the starting point for my future doctoral dissertation which I defended in Novosibirsk.' His first meeting with Cohn occurred in 1966 at the Moscow Congress, where Bokut heard him and S. A. Amitsur speak of work by their students, Andrew Bowtell and Avraham Klein, which was rather closely related to that of Bokut, who had followed a different approach. 'The Congress was the start of my friendship with Cohn', he says, one that lasted for the remainder of his life. He continues that 'in March–April of 1970 I visited Cohn at Bedford College for two months. P. M. Cohn showed great attention and concern for me' and 'organised a colloquium which many attended at which I spoke of my results and on a very new solution by Matiyasevich of Hilbert's 10th problem (George Bergman translated my talk). Cohn held a big reception for me at his home, where I chatted with the recent Fields Medalist, Paul Cohen.' Later 'I went with Cohn in his car to York, where I met Abraham Robinson and was introduced to Serre by Cohn. In general I was charmed by Cohn, his attention to me, his quality as a person, his mathematical breadth. I became a lifelong admirer of Cohn, his mathematics and his personality'. Finally he writes, 'I often mention Cohn in my seminars and lectures in Russia and China as an exceptional scholar and person, the intellectual father of mathematical constructions used by leading world mathematicians.'

4. The London Mathematical Society and other mathematical activities

Paul Cohn was dedicated to the LMS, and indeed it may be said to be part of his union with England and the UK. J. J. O'Connor and E. F. Robertson have written, in their biography of Paul (http://www-history.mcs.st-andrews.ac.uk/Biographies/Cohn.html), that 'Cohn was an enthusiastic member of the LMS, and he has served as its secretary during 1965–67, as a Council member in 1968–71, 1972–75 and 1979–84, being President of the Society during 1982–84. He also acted as editor of the LMS Monographs during 1968–77 and again 1980–93.'

Indeed, Paul may be said to have been the backbone of the Monograph series, because he was one of the Editors (with Harry Reuter) who started the series; Paul acted as Editor for two periods totalling 22 years.

Anthony Watkinson, who was at Academic Press from 1971 and then at Oxford University Press, was always associated with the LMS publishing. When Paul 'retired reluctantly' from his first stint as Editor, Watkinson also was reluctant to see him leave; but leave he did, although he returned some three years later. Watkinson says, 'the reason why I wanted to continue with Paul was that he was one of the best series editors in any field that I have ever worked with.' He continues:

> We built up a close understanding and I tried myself to be active in mathematics too. I therefore offered authors to the series and Paul showed huge tact in making clear to me whether or not such

> and such a mathematician should be encouraged and another should not be encouraged, without ever saying that the person concerned was a poor mathematician.

On the positive side, Watkinson says 'a fair number of mathematicians will have had cause to thank him. I would like to say that I brought John Conway to the series with "On Numbers and Games", but I think that Conway came to the series because of Paul.' On another but related matter he says, 'Paul helped in delicate matters and coached me in the task of bearding a journal Editor in his room, getting him to turn out his drawers and hand over manuscripts that he had been sitting on for years in some cases.' Watkinson continues: 'I remember him as a very private person though we did discuss occasionally some personal matters. I think of him as a gentleman of the old school.'

Paul served as a member of the LMS Council from 1968 to 1982; David A. Brannan recalls that as Council secretary from 1973 to 1981 he came to know Paul Cohn as 'an exceptionally nice chap'.

Alan Pears has written that when he became Meetings and Membership Secretary in 1983, Paul helped him to 'settle into this post with great kindness and courtesy'.

A few years later, Alan became secretary of the Board of Studies in Mathematics of the University of London while Paul was its chairman; Alan writes that 'shortly after my appointment both my parents died and Paul's kindness and understanding helped me through a difficult time.' This illustrates Paul's understanding from having lost his own parents at an early age, albeit in very different circumstances. Moreover, his association with the Board of Studies is indicative of the breadth of responsibilities that Paul undertook in the wider mathematical scene, including not only the University of London but also the Mathematics Committee of the Science Research Council and the Council and Committees of the Royal Society.

Part B. Research and publications, by George Bergman

5. For the non-specialist: what are non-commutative rings?

The main area of Cohn's research was non-commutative rings, and §§6–11 below, written for the reader with some knowledge of that subject, discuss his work. The present section is aimed at giving readers far from that area some idea of what the field he worked in is about.

'Ring' is a term used by mathematicians for any system of entities that can be *added*, *subtracted* and *multiplied*, and in which these operations satisfy certain laws. The most familiar example is the set of integers (whole numbers; positive and negative). The set of all real numbers, and the set of all polynomials with real coefficients, are two more examples of rings.

Let us now consider two less elementary examples, one of which will be familiar to readers who have taken a basic course in linear algebra, while the other will be described for those who have had a semester of calculus.

The first is the ring of $n \times n$ matrices over the real numbers, for a fixed $n > 1$. Note that concepts of adding, subtracting and multiplying $n \times n$ matrices are indeed defined, and again yield $n \times n$ matrices; note also that these satisfy most of the familiar laws satisfied by the similarly named concepts for numbers, for example $A(B + C) = AB + AC$. But there is one law that holds for numbers but not for matrices: commutativity of multiplication, $ab = ba$. For two $n \times n$ matrices A and B, one typically has $AB \neq BA$.

As our other example, let us consider certain *operations* on polynomials $f(x)$, namely *multiplication by* x, which, as an operation, we will call X, and *differentiation with respect to* x, generally written d/dx, which we shall here write D. We can define the 'product' of two operations as the result of applying first one and then the other. So, for example, DX is the operation that takes any polynomial $f(x)$ to $D(X(f(x)))$; that is, $\mathrm{d}/\mathrm{d}x\,(xf(x))$, and XD takes $f(x)$ to $X(D(f(x))) = x(\mathrm{d}/\mathrm{d}x\, f(x))$. Addition and subtraction are defined more mundanely; for instance, $D + X$ takes each $f(x)$ to $D(f(x)) + X(f(x))$. If we now allow ourselves to form all the operations that can be obtained from these two operations X and D, and the real numbers (where each real number r is regarded as the operation of multiplying polynomials by r), using repeated applications of addition, subtraction and multiplication, we get what is called the 'ring of differential operators generated by X and D'. Like the ring of $n \times n$ matrices, this ring fails to satisfy the commutative law of multiplication. Indeed, using the product law for differentiation from calculus, the reader can check that $DX - XD = 1$. (Incidentally, in the magical realm of quantum mechanics, it is this instance of non-commutativity of differential operators that yields the Heisenberg uncertainty principle, saying that one cannot exactly measure both the position and the velocity of a particle simultaneously.)

This is not the place to give the full list of conditions defining a ring. Suffice it to say that the most studied class of these objects, called *commutative rings*, satisfy a list of laws that includes commutativity of multiplication, $ab = ba$, whereas the main area of Cohn's research was *non-commutative rings*, rings for which the other main conditions are assumed, but not that one; and which thus include the two examples just noted.

Going back to the commutative ring we began with: at some time in our childhood, we learn the structure of the ring of integers; not long afterwards, we learn about rational numbers (fractions). But in learning about the latter, we do not start from scratch: we learn that each fraction is obtained by dividing one integer by another. Much later, we learn that by dividing one polynomial by another, we similarly get things called *rational functions*. Commutative rings such as the ring of the rational numbers, and the ring of rational functions in x and y, in which one can *divide* any element by any nonzero element, are called *fields*. Based on the way in which rational numbers are constructed from integers, the field of rational numbers is called the *field of quotients* of the ring of integers; and the field of rational functions is likewise the field of quotients of the polynomial ring.

Not every commutative ring has a field of quotients. For instance, if one starts with polynomial functions on the plane (polynomials in two variables) and then restricts the set on which one considers them from the whole plane to the union of the x axis and the y axis, one finds that, for these restricted polynomial functions, the function x and the function y, though neither is the zero function, have for product the zero function, and this throws a monkey-wrench into any attempt to form a field of quotients in which both x and y can occur as denominators.

But the answers to the questions of *which* rings have fields of quotients, and how to compute with these, have long been understood, and are second nature to any algebraist.

Not so for non-commutative rings! But—as described for the specialist in the next few sections—a major part (although far from all) of Paul's work was to advance enormously our understanding of that subject.

(We remark that although the term 'non-commutative rings' is used, as above, when contrasting the subject with that of commutative rings, the subject does not exclude rings that are commutative, but merely includes them along with those that are not; workers in the field often call the objects of their study 'associative rings', alluding to the identity $a(bc) = (ab)c$, which *is* still assumed. Both terms are used in the sections below, depending on the point being made.)

6. The problem: mapping rings into division rings

In surveying Paul's research, we will begin with his stunning achievement, the classification of homomorphisms from a fixed associative ring R into division rings* D, then move backwards and forwards from there. To see what he gave us, we need to review some background.

For a commutative ring R, the classification of homomorphisms into fields is well understood. The image of R under a homomorphism into a field F is an integral domain, hence isomorphic to R/P for some prime ideal $P \subseteq R$. The subfield of F generated by the image of R can thus be identified with the field of quotients $Q(R/P)$. Hence, given R, the isomorphism classes of pairs (F, f), where F is a field and $f : R \to F$ a ring homomorphism such that $f(R)$ generates F as a field, correspond bijectively to the prime ideals of R.

In this classical situation, if we take for R a polynomial ring $k[X_1, \ldots, X_n]$ over a field k, and take for P the zero ideal, we get the rational function field $k(X_1, \ldots, X_n)$. We would like to say that this is an extension of k as a field by a 'universal' n-tuple of elements, but what universal property can these elements have? Given $a_1, \ldots, a_n$ in an arbitrary field extension F of k, there will not, in general, exist a homomorphism $k(X_1, \ldots, X_n) \to F$ carrying each X_i to a_i, since field homomorphisms are one-to-one.

Yet every element of $k(a_1, \ldots, a_n)$ can clearly be obtained by substituting $a_1, \ldots, a_n$ for $X_1, \ldots, X_n$ in some element of $k(X_1, \ldots, X_n)$. If we analyse this fact, we discover that $k(a_1, \ldots, a_n)$ is a homomorphic image of a *local subring* of $k(X_1, \ldots, X_n)$, consisting of those rational functions that can be written with denominators that do not vanish under the indicated substitution. A homomorphism φ from a local subring of a field E to another field F is called a *specialization* from E to F. These are trickier to study than homomorphisms from E to F, because E alone does not determine the domain of the map φ; but once one sets up the right definitions, one finds that $k(X_1, \ldots, X_n)$ is indeed generated over k by an n-tuple $X_1, \ldots, X_n$ universal with respect to specializations.

None of the above elegant theory appears to be applicable to non-commutative rings. A non-commutative ring R without zero divisors need not be embeddable in a division ring; and if it is embeddable, the structure of the division ring generated by its image is not in general unique. The first fact means that the possible kernels of homomorphisms f from a fixed ring R into division rings D form some non-obvious subclass of the completely prime ideals (the ideals P such that R/P has no zero divisors); the second means that when D is a division ring generated by the image of a homomorphism f on R, the kernel of f may not be enough to determine the pair (D, f) up to isomorphism. For an explicit example of this

* Paul used the term 'field' in the context of non-commutative ring theory to mean 'division ring'. Terms used by others have included 'skew field' and its abbreviation 'sfield'. Below we will use the term 'division ring', reserving 'field' for the commutative concept.

We will also, for brevity, be loose about questions of which conditions have distinct right and left forms; e.g., we will not mention, when we introduce the properties of being a fir and a semifir in §§8 and 9, that for the former there are distinct 'left fir' and 'right fir' conditions, while the latter is right–left symmetric.

second fact, let us recall that the free monoid $\langle X, Y\rangle$ on two generators has many embeddings in groups, and let us compare two of these: (i) its canonical embedding in the free group on X and Y, and (ii) the map into the group of invertible affine transformations of the real line, $\{a_{m,b} : t \longmapsto mt + b \mid m \neq 0\}$, given by $X \longmapsto x = a_{1/2,\,0}$, $Y \longmapsto y = a_{1/2,\,1/2}$. (Key to showing that this map is an embedding: examine the relation between a monoid word $w(X, Y)$ and the interval $w(x, y)([0,1]) \subseteq [0,1]$.) These monoid embeddings induce embeddings of the monoid algebra $k\langle X, Y\rangle$ over any field k, that is, the free associative algebra on X and Y, into the group algebras of these two groups. As a consequence of the orderability of these groups, their group algebras can each be embedded in a Mal'cev–Neumann division ring of formal power series (see Mal'cev (1948) and Neumann (1949)). Now the elements x and y used in (ii) have the property that xy^{-1} and $y^{-1}x$ commute (since they are both translations), while in the free group on X and Y used in (i), XY^{-1} and $Y^{-1}X$ do not commute; so the two pairs (D, f) are not isomorphic, although both maps f are injective.

What about the question of finding a 'universal' division algebra over k on an n-tuple of elements $X_1, \ldots, X_n$? That is, among embeddings of the free associative algebra $k\langle X_1, \ldots, X_n\rangle$ in division algebras, is there one that is universal with respect to specialization? S. A. Amitsur in fact proved (Amitsur 1966) that there exists a division ring having the desired universal property, which he constructed by using an ultraproduct of division rings generated by generic matrices of unbounded integer sizes over commutative rings. But is there any natural way to obtain this division ring from $k\langle X_1, \ldots, X_n\rangle$? I had played with that question, but concluded that it was a hopeless dream—a dream so beautiful that it was worth trying one's hand at from time to time, but not something one should expect to achieve.

And then Paul astonished us all by solving the problem of classifying homomorphisms of a ring R into division rings, and did this in a way that showed in particular which division ring was universal for $R = k\langle X_1, \ldots, X_n\rangle$.

7. Prime matrix ideals

The key insight of his approach (which he says in the notes to Chapter 7 of (14) was inspired by work of Schützenberger and Nivat on rational non-commuting formal power series) is that to study a homomorphism $f : R \longmapsto D$, one should look not merely at the set of elements of R that go to zero under f, but also at the set of square matrices over R that become singular under f, which he named the *singular kernel* of f.

He showed that the singular kernel of f precisely determines the rational relations satisfied by the images of the elements of R in D; he found necessary and sufficient conditions for a set P of square matrices over an arbitrary associative ring R to be such a singular kernel, calling a set with these properties a 'prime matrix ideal' of R; and he obtained an explicit construction for the division subring of D generated by $f(R)$ in terms of that singular kernel.

He showed, moreover, that *inclusions* of prime matrix ideals correspond to *specializations* over R between the corresponding division rings; and for an important class of rings R, the *free ideal rings* (whose definition we shall recall in the next section), which include the free associative algebras $k\langle X_i \mid i \in I\rangle$, he showed that the class of prime matrix ideals has a smallest member, the set of all square matrices A that can be factored $A = BC$ where B is $d \times (d-1)$ and C is $(d-1) \times d$. Thus, the division ring corresponding to that prime matrix ideal has the desired universal property.

For another class of examples to which these results were immediately applicable, consider any two rings R_1 and R_2 with a common subring R_0. One can form their coproduct $R_1 *_{R_0} R_2$ with amalgamation of R_0, and Paul had previously shown that if R_0 was a division ring D_0, and R_1 and R_2 were free ideal rings, then this coproduct was again a free ideal ring. Hence this is true, *a fortiori*, when all three objects are division rings, D_0, D_1 and D_2; and the universal division ring associated with the least prime matrix ideal of the free ideal ring $D_1 *_{D_0} D_2$ gives a 'coproduct of D_1 and D_2 over D_0 as division rings', $D_1 \circ_{D_0} D_2$, again characterized as universal with respect to specialization.

Paul obtained the above results in about 1970, announced them in 1971 in (12), and gave the detailed proofs in (13) and (15).

8. Free ideal rings …

We stated, above, an elegant description of the least prime matrix ideal of R in the case where R is a free ideal ring.

What is a 'free ideal ring'?

That concept had developed out of Paul's earlier work on free associative algebras $k\langle X_i \mid i \in I\rangle$ (k a field). When $|I| = 1$, this algebra $k\langle X\rangle$ is also the free commutative associative algebra; that is, the polynomial algebra $k[X]$, which by the division algorithm for polynomials is a principal ideal domain. As soon as $|I| = 2$, however, both the polynomial ring $k[X, Y]$ and the free associative algebra $k\langle X, Y\rangle$ lose the property that ideals, respectively left ideals, are principal; it is easy to see that neither the ideal of the first nor the left ideal of the second generated by X and Y can be generated by a single element. In the case of $k\langle X, Y\rangle$, those generators are in fact left linearly independent over the base ring, which makes the ring particularly bad from the classical point of view. Rings without zero divisors in which any two nonzero elements have a nonzero common left multiple are the *left Ore* rings, the rings for which the classical construction of division rings of fractions is possible; so the existence of elements without such a left common multiple puts free associative algebras 'beyond the pale'. Yet looking at things another way, the fact that X and Y are left linearly independent means that the left ideal that they generate is a *free* left module, and in that respect, it resembles ideals of $k[X]$ better than the ideal of $k[X, Y]$ generated by X and Y.

Of course, not every pair of nonzero elements of $k\langle X, Y\rangle$ is left linearly independent. Obvious counterexamples are pairs of elements $f(u)$ and $g(u)$, where u is any element of that ring, and f and g are polynomials in one indeterminate over k, since they satisfy $f(u)g(u) = g(u)f(u)$. A less obvious example is given by the elements $YX + 1$ and X, which have the common left multiple $XYX + X$. But in both these cases, one finds that the left ideal generated by our two elements is free on *one* generator. (For the $f(u), g(u)$ case, one can deduce this by applying the division algorithm to the polynomials $f(t), g(t) \in k[t]$; in the other case, we see that $1 \cdot (YX + 1) - Y \cdot X = 1$, so that $YX + 1$ and X generate $R \cdot 1$ as a left ideal.)

And, to make a long story short, every left ideal of $k\langle X, Y\rangle$ (and more generally, of the free associative algebra on any set, finite or infinite, over a field) is free on some set of generators.

Over a non-commutative ring, even the classification of free modules can be messy, since such a module may have free bases of various cardinalities; but this does not happen over our free associative algebras. (It is automatically excluded over any ring admitting a homomorphism to a field.) Paul gave rings with this combination of properties—that all left ideals are free, and all free modules have invariant basis number—the name *free ideal rings*, or *firs* (6).

(L. A. Skornjakov of Moscow wrote a paper (1965) in which he renamed firs *konovskije kol'ca* ['Cohn rings']. Paul was upset: 'But then what will I call them?' Fortunately for him, his term 'fir' prevailed.)

How does one prove a ring to be a fir? One sufficient condition that Paul established is an elegant generalization of the division algorithm for polynomials. Recall that that algorithm says that given two elements $a, b \in k[X]$ with $b \neq 0$, we can, by subtracting a left multiple of b from a, reduce its degree to less than that of b. The modified condition, for two elements a and b of a ring with a degree function v, says roughly that one can do the same *if* the elements a and b are left linearly dependent. More precisely, rather than restricting to the case where a and b have an actual left linear dependence relation, one merely assumes that the sum of some left multiple of a and some left multiple of b has less than the expected degree (that they are 'left v-dependent'), and concludes that the greater of the degrees of a and b can be reduced by subtracting from the element of that degree an appropriate left multiple of the other element. This property is actually a statement about elements-modulo-elements-of-lower-degree (technically: elements of the associated graded ring). Rather than just assuming that condition for pairs of elements, Paul's condition assumes the corresponding statement for arbitrary finite families, so that one can handle left ideals generated by more than two elements.

Because of the added v-dependence hypothesis, Paul named his condition 'the weak algorithm' (5). In retrospect, the term is excessively self-effacing. Although the condition can be looked at as a weakening of the classical division algorithm by the addition of an extra hypothesis, that extra hypothesis is vacuous in the commutative case, while in the non-commutative case it is what we need if the algorithm is not to force our rings to be Ore rings; that is, next-door to commutative. So the 'weak algorithm' is not really weak. But the name (like 'imaginary number') has become standard.

Workers in commutative ring theory, and its partners algebraic geometry and number theory, far outnumber those in non-commutative rings, and tend to regard the latter area as excessively 'far out'. ('If we don't know our rings are commutative, how can we trust anything we know?') Among non-commutative ring theorists there is a tension between the tendencies to hug close to the border with the more popular commutative theory, judging results as 'good' to the extent that they look like standard results from the commutative case, and to venture far from the commutative and discover what results are natural to the rings one finds there. Paul was one who strode into the wilds of the non-commutative, and uncovered great beauty.

The above dichotomy between mimicking the commutative world and leaving it behind is, of course, an oversimplification. Indeed, the way the above discussion introduced the concepts of fir and weak algorithm shows that what happens deep in the world of the non-commutative may be describable by a creative extension of what is known in the commutative case.

Another such creative generalization concerns the concept of torsion module. Over a commutative principal ideal domain R, one knows exactly what the finitely generated torsion modules look like: they are direct sums of modules R/Rq, where q is a power of a nonzero irreducible element of R. When R is a non-commutative fir, the first question we must decide is what class of modules to focus on. Given a nonzero element $a \in R$, there will typically be elements $b \in R$ that are left linearly independent of a, in which case the image of b in R/Ra will not be a torsion element. So: should we look at some class of modules typified by the modules R/Ra ($a \neq 0$), or at a class of modules all of whose elements are torsion? Paul discovered that one obtains a beautiful theory if one makes the former choice and studies finitely generated left modules in which the number of relators equals the number of generators, in a robust way;

precisely, modules $M = R^d/R^dA$, where A is a $d \times d$ matrix that *cannot* be factored $A = BC$, where B is $d \times (d - 1)$ and C is $(d - 1) \times d$. Over a commutative principal ideal domain, the modules with such presentations are the finitely generated torsion modules. Paul called such left modules M over a fir *left torsion modules* (10), and the analogous class of right modules the *right torsion modules*; he showed that each of these classes forms an abelian category, and that there is a duality (contravariant equivalence) between the two categories. The existence, within torsion modules, of non-torsion submodules, noted above, turns out to be immaterial: one cannot 'see' such a submodule from within the category because it is never the kernel or image of a homomorphism of torsion modules.

We do not yet have the understanding of torsion modules over general firs that we do for commutative principal ideals domains. In the latter case, the minimal building blocks are the modules R/Rp for p an irreducible element of R, and these fit together in easily understood ways. In the general case, more exploration is needed.

The module theory of firs turned out to be providential as background for Paul's construction of universal maps into division rings, and it is tempting to conjecture that this work in the years preceding 1970 was aimed at providing that background—tempting, but unlikely. It is hard to imagine that, before discovering the relevant properties of free algebras, one could predict what use they could be put to. And in working with Paul on firs, free algebras, and so on, from 1966 onwards, I heard no foreshadowing of this idea.

Still, the preface of the first edition of (14) begins with the quotation from *A Midsummer Night's Dream*,

> I have had a dream,—
>   past the wit of man to say what dream it was:
> man is but an ass,
>   if he go about to expound his dreams.

So who knows what dream he may have been keeping to himself?

9. … AND THEIR RELATIVES

Paul brought together in his 1971 book (14) the main results that had been obtained so far in this area. Its title, *Free rings and their relations*, is a multiple play on words.

A free object is, by definition, presented by generators subject to *no relations*; so that title is, on the face of it, an oxymoron. However, elements other than the free generators can satisfy non-trivial relations—we noted, for instance, the relation $X \cdot (YX + 1) = (XY + 1) \cdot X$—and the study of such relations is, in one form or another, what much of the theory of free algebras is about.

'Relations' also means 'relatives', and it would be a pity to prove results for free rings alone, without looking at larger classes of rings to which the same or similar methods apply. We have already seen that free algebras fall within the class of rings with weak algorithm, and these within the class of firs. Other examples of firs include group algebras of free groups, and, as we have mentioned, ring-coproducts of division rings.

Recall that whereas the classical division algorithm concerns pairs of elements, the statement of the weak algorithm refers to arbitrary finite families. If we only assume the condition of that algorithm for families of $\leq n$ elements, for a given n, we have what Paul named the 'n-term weak algorithm', yielding rings in which every left ideal generated by $\leq n$ elements is free of unique rank; these he named *n-firs*. He showed in particular that, for various sorts of rings arising from *universal matrix constructions*, if those constructions are such that, whenever a matrix with r columns is multiplied by a matrix with r rows in one of the imposed relations, we have $r > n$, then the resulting ring will satisfy the n-term weak algorithm (8).

There are also rings that are n-firs for every positive integer n, and are thus called *semifirs*, but which are not firs. Such examples cannot be established directly by the weak algorithm, because if a degree function v satisfies the relevant condition for all finite n, the ring is a fir. But one can get examples as direct limits of firs (for example, $k[X, X^{1/2}, \ldots, X^{2^{-n}}, \ldots]$ is a direct limit of polynomial rings $k[t]$) or by ultraproduct constructions.

It is a general principle of module theory that whatever free modules are good for, *projective* modules are likely to be equally good for; so important relatives of firs and semifirs are rings all of whose left ideals (respectively finitely generated left ideals) are projective as modules. These are the left (*semi*)*hereditary* rings. To make these conditions comparable to those of being a (semi)fir, one needs some analogue of the condition that free modules have unique rank. One such condition is the existence of a 'rank' function from isomorphism classes of finitely generated projective modules to natural numbers (or non-negative rationals, or reals) under which the rank of a direct sum is the sum of the ranks of the summands, and Paul also studied rings of these sorts.

Finally, although rings like $k[X, Y]$ are homologically 'worse' than rings like $k[X]$ and $k\langle X, Y\rangle$, they still have good qualities. Paul investigated a class of non-commutative rings that are as well-behaved as $k[X, Y]$ and better than $k[X, Y, Z]$, namely the rings over which Sylvester's law of nullity holds, a matrix-theoretic property that has a key role in the construction of the universal division rings of firs. He named these *Sylvester domains*. They include all free associative algebras over commutative principal ideal domains; for example $\mathbb{Z}\langle X, Y\rangle$.

A different sort of 'relatives' of free algebras are non-commuting formal power-series algebras. To say that the relation between a free associative algebra $k\langle X_i \mid i \in I\rangle$ and the formal power-series algebra $k\langle\langle X_i \mid i \in I\rangle\rangle$ is that the latter is a completion of the former is correct, but it is not the most relevant fact for the point at hand. Recall that in studying a free associative algebra, one uses the highest-degree terms of elements to define one's degree function, and uses finite induction to take advantage of the weak algorithm. Elements of a power-series algebra have no highest-degree term, so one looks instead at *lowest*-degree terms. In the non-commutative formal power-series case, an analogue of the weak algorithm allows one to take left linearly dependent elements, and to use left linear maps to strip off more and more low-degree terms from one of them, and finally use completeness to conclude that it is a linear combination of the others. The graded algebras associated with the highest-degree-term filtration in the free algebra case, and with the lowest-degree-term filtration in the formal power-series case, are the same; and the property of those graded algebras that gives the former algebras the weak algorithm, gives the latter the analogous construction sketched above, which Paul named the *inverse weak algorithm*. He showed that a complete filtered ring with inverse weak algorithm is a semifir and is a 'topological fir', in the sense that every one-sided ideal has a linearly independent topological generating set.

10. Earlier work

Paul's 1951 doctoral thesis (1), written under the supervision of Philip Hall, concerned the relation between free groups, free associative algebras and free Lie algebras, and his first two published papers were based on that thesis.

Although the subject of free associative algebras points towards his later work, his next few papers moved in various directions with no common theme. Four of them answered some unrelated questions, two posed by B. H. Neumann and two by I. Kaplansky (three counter-examples and a proof all together). There were also two papers on special and semispecial Jordan algebras, two on pseudovaluations on commutative rings, two on rings where all (or almost all) equations $a\xi - \xi b = c$ $(a, b \neq 0)$ have solutions, two on embeddings of semigroups, and two expository volumes, one on Lie groups and one on linear equations, as well as single papers on several other topics.

Amid these, in 1959 and 1960, came two papers on *free products* of associative rings, the name that Paul used for coproducts of rings over a common subring when they have the properties that the given rings embed faithfully in the coproduct, and are disjoint except for the common subring. The first of these papers concerned conditions for such free products to exist (that is, for the coproduct to have these properties) in terms of module-theoretic properties of the given system of rings; the second showed that the coproduct of two division rings over a common division subring is (in the language he would later use, noted in the preceding section) a 2-fir. Neither of these results was needed as such in his later work: the free products that he considered later were almost always over division rings, so that the delicate module-theoretic considerations of the 1959 paper were not needed, while the 2-fir result of the 1960 paper was to be subsumed in the statement that such a free product is a fir. But these were steps towards that body of work. The year 1961 saw the first major result in that work, the statement of the weak algorithm (not yet so named) and its consequences, in (2).

Two other 1961 papers of Paul's are also worth mentioning:

On the one hand, in (4) he studied extensions K/k of division rings such that K is 2-dimensional as a *right* k-vector-space. That topic is much less trivial than in the commutative case. In particular, he obtained an example in which the left dimension was not the same as the right dimension, answering a long-standing open question of E. Artin.

In (3), on the other hand, he proved the embeddability of a large class of rings into division rings, not by a construction anything like his later universal one, but under the assumption that the ring has a filtration 'modulo which' it behaves approximately like a right Ore ring. This applies in particular to the universal enveloping algebra of any Lie algebra over a field, because by the Poincaré–Birkhoff–Witt theorem, such a ring has a filtration whose associated graded ring is a commutative polynomial ring. Thus, as he observed, the result is applicable to free associative algebras, regarded as universal enveloping algebras of free Lie algebras. It is striking how dissimilar this filtration is from the one used in his later results, the standard filtration on a free associative algebra.

That free associative algebras could be embedded in division algebras was not new (it had been proved by Moufang (1937), Mal'cev (1948) and Neumann (1949)). But this paper suggests that ways of getting such embeddings, and perhaps the question of whether there were 'canonical' embeddings to be found, may already have been on Paul's mind.

Incidentally, a few years after Paul obtained his universal embedding of $k\langle X_1, \ldots, X_n\rangle$ in a division algebra, Lewin (1974) showed that, in any Mal'cev–Neumann division algebra on the free group on $X_1, \ldots, X_n$, the division subalgebra generated by $X_1, \ldots, X_n$ is isomorphic to Paul's universal division algebra.

11. Later work

Paul's study of universal fields of fractions did not end with the proof of their existence. Just as an element u of the field of fractions of a commutative ring has various expressions ab^{-1}, and one studies the relation between such expressions (finding, for instance, that if the ring is a unique factorization domain, there is an essentially unique expression for u 'in lowest terms'), so, likewise, given an element u of a division ring D generated by an image of a non-commutative ring R, one may look at various expressions for u in terms of inverses of square matrices over R (say as products $x^{\mathrm{T}}A^{-1}y$, where A is an $n \times n$ matrix and x and y are height-n vectors over R—this is one of several closely related forms that Paul studied) and seek 'lowest terms' expressions and other invariants of u. One such invariant, which always has the value 1 in the commutative case but is unbounded in general, is the least n for which one can get such an expression $u = x^{\mathrm{T}}A^{-1}y$, which Paul named the 'depth' of u. He obtained striking results on these topics in (18–20).

If S is a generating set for the above ring R as an algebra over a field k (for instance, if R is the free associative algebra on S), and if one has an expression for an element $u \in D$ using the inverse of a matrix over R as above, then one can construct another such expression for u, in which all the matrix entries are k-linear combinations of 1 and elements of S, at the price of using larger matrices. Properties of such expressions are developed in (24).

Incidentally, ring extensions obtained by inverting matrices are a powerful tool even when the result is not used to construct a skew field, and these are now often known as 'Cohn localizations'; see, for example, Ranicki (2006).

The papers on miscellaneous questions in algebra noted in the preceding section did not come to a halt when Paul began obtaining his central results. If, in his early years, they suggested someone who had not yet found his direction, their continuation indicates that the edifice he was creating did not extinguish his interest in the rest of algebra. Several papers on radical rings, and on general linear groups of commutative rings, are in this category. Of course, one cannot always draw a sharp line separating these from papers in his main area. For example, in (9) he studied the groups GL_2 for many sorts of rings, ranging from classical commutative rings of algebraic integers to free associative algebras. Others of his papers concern properties of division rings obtained by Ore's construction, and so could be considered either as cases of his general theory of division rings or as a visit to a classical topic.

Another important thread in his work, beginning early on and continuing throughout his career, is the factorization of ring elements. His early paper (11) on *commutative* rings with various factorization properties is still regularly cited. The equation $X \cdot (YX + 1) = (XY + 1) \cdot X$ in the free associative algebra $k\langle X, Y\rangle$ (mentioned twice already) might seem to indicate that that ring satisfies nothing like unique factorization. But after one has absorbed Chapter 4 of (29), one can view the theory of commutative UFDs as a degenerate case of the much more diverse theory of not-necessarily-commutative rings with *distributive divisor lattices*, which include the free associative algebras.

Although we suggested that the quote from *A Midsummer Night's Dream* that introduced the first edition of (14) might refer to the dream that was achieved in that book, it could have referred to a grander dream. Some of Paul's later expository articles, such as (16) and (21), look towards the possibility of a full-blown non-commutative algebraic geometry. An important tool in algebraic geometry is the theory of valuations on fields. Valuations on division rings were the topic of the doctoral thesis of one of Paul's students; the two wrote jointly on that topic in (17), and Paul wrote three further papers on the subject.

The question of whether a non-commutative algebraic geometry based on homomorphisms into division rings can in fact be achieved—whether, indeed, it is what we should be looking for—remains to be answered.

12. Textbooks, translations, etc.

If one searches MathSciNet for *books* by Paul, one obtains 25 results. If one cuts this down as far as possible, by treating multivolume works as single items, and by treating as extensions of a work its translations into other languages and its subsequent editions (even if the latter have been considerably revised, and even if they appeared, at the wish of the publisher, under changed titles), one can bring the number down to about 10, to which a web search adds three books not shown on MathSciNet. (He also wrote some mathematical articles for *Encyclopaedia Britannica*, although that work does not show authors' names, so that these articles do not appear in the online bibliography associated with this memoir.)

Two of Paul's books—(14) (revised as (29)) and (23)—are presentations of the central areas of his work: free algebras, free ideal rings, and constructions of division algebras.

Another advanced monograph (7) (which, with its second edition and Russian translation, constitutes three of the MathSciNet listings) is not specific to his area of research, but treats some basic material underlying most of algebra, and the name P. M. Cohn is probably most widely known among mathematicians outside ring theory for that book.

Other advanced monographs include the book on Lie groups mentioned in §10, the one on GL_2 of rings mentioned in §11, a volume on Morita equivalence, and one on algebraic numbers and algebraic functions. At a more elementary level are the notes on linear equations mentioned in §10, a text on linear algebra, one on solid geometry, and a basic graduate text in ring theory.

Finally, Paul was the author of a multi-volume textbook on algebra, which begins with the material of an undergraduate 'abstract algebra' course, covers the graduate 'groups, rings and fields' course, and goes on to more advanced topics. At the wishes of the publishers, the successive revisions of this work several times changed title and number of volumes (it constitutes 7 = 2 + 3 + 2 MathSciNet citations, and there was a retitled version of one volume not shown by MathSciNet). Its final form comprises two volumes (26, 27).

Paul was comfortable enough with French and German to publish seven papers in the former language and three in the latter, and sufficiently proficient in mathematical Russian to translate two lengthy articles from Russian for the *Encyclopaedia of Mathematical Sciences* in the mid 1990s. He also co-translated from the French, with J. Howie, a volume by Bourbaki.

(He did at least one piece of non-mathematical translation work, mentioned to me in an email of September 1999, in the course of describing the things he was busy with: 'Rather foolishly I took on the job of translating a book on genealogy from … German to English, but that is also finished now. It was extremely interesting, at least from the linguistic point of view, and not always easy.' But I have not been able to find any reference to a book on genealogy listing him as translator.)

Figure 3. Paul Cohn on a walk near to the Weisshorn in Switzerland, March 1994 or 1996. (Online version in colour.)

13. 'It is not your duty to complete the work'

When one opens the second edition of (14), one finds that the *I have had a dream* quotation has been replaced by one from Rabbi Tarphon (first century CE):

> It is not your duty to complete the work,—
> But neither are you free to desist from it.

This was used again in (29), the final version of (14), on which Paul worked from 1999 to 2004, and which came out shortly after his death.

He is at last free to desist from his work. Let us hope others will carry it further.

Part C. Conclusion, by George Bergman and Trevor Stuart

14. Summary and appreciation

Paul Cohn's achievements in non-commutative ring theory are ones of which he could feel justifiably proud. Moreover his books contributed greatly to algebraic knowledge in the mathematical community, both at the research level and in undergraduate texts. He was greatly revered for these reasons.

His quiet personality, coupled with the ability to listen and respond, was appreciated by students and other researchers alike. He was respected and admired the world over for these qualities. Paul Cohn gave great support to the LMS and was its President from 1982 to 1984.

Paul Cohn was a loving family man who never forgot his background in Germany and who always remembered his parents' sacrifice and devotion in sending him by Kindertransport to England. He cherished his UK citizenship, and England was his home. He had a great love of all activity, including walking in the Alps (figure 3). Moreover he loved mathematics, to which he contributed greatly.

Honours and awards

1972	Lester R. Ford Award, Mathematical Association of America
1974	Senior Berwick Prize, London Mathematical Society
1980	Fellow of the Royal Society
1982–84	President of the London Mathematical Society
1986	Astor Professor of Mathematics at University College London

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