A p-ADIC GEOMETRIC LANGLANDS CORRESPONDENCE FOR GL_1

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ABSTRACT. The (de Rham) geometric Langlands correspondence for GL_n asserts that to an irreducible rank n integrable connection on a complex smooth projective curve X, we may naturally associate a D-module on $\mathrm{Bun}_n(X)$, the moduli stack of rank n vector bundles on X. Making appropriate changes to the formulation there are also Betti and l-adic versions of the above correspondence. We conjecture the existence of a p-adic geometric Langlands correspondence relating irreducible, rank n, F-isocrystals on X (now a curve over $\mathbb{F}_p$) to arithmetic D-modules on $\mathrm{Bun}_n(X)$. We prove the conjecture in the case $n = 1$.

1. INTRODUCTION

Let X be a smooth, projective curve over $\mathrm{Spec}(\mathbb{F})$, where $\mathbb{F}$ is a finite field of characteristic p. Let $\mathcal{K}$ be the field of rational functions on X and $\mathbb{A}_{\mathcal{K}}$ denote the ring of adeles of $\mathcal{K}$. Let $\mathcal{K}^{nr}$ be the maximal unramified extension of $\mathcal{K}$, contained in some fixed separable closure. Let ℓ be a rational prime distinct from p. An *unramifed* ℓ-adic Galois represention is a continuous representations:

$$\rho : Gal(\mathcal{K}^{nr}/\mathcal{K}) \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell),$$

whose determinant has finite image.

The classical *unramified* Langlands correspondences for GL_n (over function fields) asserts that there is a *natural* map of sets:

$$\left\{ \begin{array}{c} \text{Isomorphism classes of irreducible,} \\ \text{unramified, } n\text{-dimensional,} \\ \ell\text{-adic Galois representations} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Isomorphism classes of unramified,} \\ \text{cuspidal automorphic representations} \\ \text{of } GL_n(\mathbb{A}_{\mathcal{K}}) \end{array} \right\}$$

By natural we mean that both sides must be compatible in some precise sense involving the classical Hecke operators.

It was Drinfeld and Laumon who first observed that it was possible to reinterpret each side of this correspondence in a more geometric way, using Grothendieck's theory of ℓ-adic

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sheaves. This ultimately led to an ℓ-adic geometric Langlands correspondence:

$$\begin{array}{ccc}\left\{\begin{array}{c}\text{Isomorphism classes of irreducible,}\\ n\text{-dimensional, } \textit{lisse}\\ \ell\text{-adic sheaves on } X\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\text{Isomorphism classes of } \ell\textit{-adic}\\ \text{perverse sheaves on } \mathrm{Bun}_n(X), \text{ the moduli}\\ \text{stack of rank } n \text{ vector bundles on } X\end{array}\right\}\\ E & \mapsto & \mathrm{Aut}(E)\end{array}$$

The Hecke compatibility in this situation comes from geometrically re-interpretting the Hecke operators. We say that $\mathrm{Aut}(E)$ is a Hecke eigensheaf for E (§3.8 [4]).

In this geometric setting both sides of the correspondence make sense even if X is a smooth, projective curve over $\mathrm{Spec}(\mathbb{C})$. By applying the usual comparison between ℓ-adic and Betti cohomology we obtain the *Betti* geometric Langlands correspondence:

$$\left\{\begin{array}{c}\text{Isomorphism classes of irreducible,}\\ n\text{-dimensional,}\\ \text{complex local systems on } X\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{complex perverse sheaves}\\ \text{on } \mathrm{Bun}_n(X)\end{array}\right\}$$

Finally we may go even further and apply Riemann-Hilbert correspondence, to give the *de Rham* geometric Langlands correspondence:

$$\left\{\begin{array}{c}\text{Isomorphism classes of irreducible,}\\ n\text{-dimensional,}\\ \text{integrable connections on } X\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{regular, holonomic } \mathcal{D}\text{-modules}\\ \text{on } \mathrm{Bun}_n(X)\end{array}\right\}$$

All of these correspondences are now theorems thanks to the work of many mathematicians, including Drinfeld, Lafforgue, Frenkel, Laumon, Gaitsgory, et al.

In the case $n = 1$ all these correspondences greatly simplifiy. The moduli of rank 1 vector bundles admits a fine moduli space, the Picard variety $\mathrm{Pic}(X)$. Here we may take $\mathrm{Aut}(E)$ to be a lisse ℓ-adic sheaf/complex local system/integrable connection on $\mathrm{Pic}(X)$. The Hecke eigensheaf property is also much easier to state in this setting. Observe that is a natural morphism

$$\begin{array}{rcl} h : X \times \mathrm{Pic}(X) & \to & \mathrm{Pic}(X)\\ (x, \mathcal{L}) & \mapsto & \mathcal{O}_X(x) \otimes \mathcal{L} := \mathcal{L}(x).\end{array}$$

The compatibility between E and $\mathrm{Aut}(E)$ is the assertion that there exists an isomorphism $E \boxtimes \mathrm{Aut}(E) \cong h^*(\mathrm{Aut}(E))$. The existence of $\mathrm{Aut}(E)$ in this case was proven by Deligne using purely geometric methods (§4.1 [4]).

We see that for Betti, ℓ-adic and de Rham Weil cohomologies we have a geometric Langlands correspondence. This paper is the first step to an extension of this program to the p-adic setting.

As above, let X be a smooth, projective curve over a finite field $\mathbb{F}$. The p-adic analogue of an integrable connection on X is a convergent F-isocrystal (from the perspective of [1]). Similarly, the p-adic analogue of a $\mathcal{D}$-module is an *arithmetic* F-$\mathcal{D}$-modules ([2]).

The p-adic Geometric Langlands Conjecture for GL_n. Let X be a smooth projective curve over $\mathbb{F}$. Let E, an absolutely irreducible, rank n, convergent F-isocrystal on X. Then there exists an arithmetic F-$\mathcal{D}$-module, $\mathrm{Aut}(E)$, on $\mathrm{Bun}_n(X)$ such that $\mathrm{Aut}(E)$ is a Hecke eigensheaf for E..

In this paper we prove this result for $n = 1$. As in the ℓ-adic, Betti and de Rham setting we will not have to work with the full category of arithmetic F-$\mathcal{D}$-modules. Our main result is the following:

p-adic Geometric Langlands for GL_1. Let $K/\mathbb{Q}_p$ be a finite extension with residue field $\mathbb{F}$. Let E be a rank one convergent F-isocrystal on X with coefficients in K. Then there exists $\mathrm{Aut}(E)$, a convergent F-isocrystal on $\mathrm{Pic}(X)$ with coefficients in K such that E is a Hecke eigensheaf for Aut_E. More precisely, such that

$$h^*(\mathrm{Aut}(E)) \cong E \boxtimes \mathrm{Aut}(E).$$

We remark that the theorem remains true (with the same proof) if we consider convergent isocrystals without Frobenius structure. Perhaps this will have some significance in the higher rank case.

2. Convergent F-Isocrystals

Let us briefly review the theory of convergent F-isocrystals from the perspective of [1].

As in the introduction, let $\mathbb{F}$ be a finite field of characteristic p. Let R be a complete DVR with residue field $\mathbb{F}$ and field of fractions K. Let Y be a separated scheme of finite type over $\mathrm{Spec}(\mathbb{F})$. Let $\mathfrak{P}$ be a flat p-adic formal R-scheme together with a closed immersion $Y \to \mathfrak{P}$, such that $\mathfrak{P}$ is formally smooth in a neighborhood containing the image of X. Let $\mathfrak{P}_{\mathrm{rig}}$ denote the generic fibre in the sense of Berthelot and Raynaud (§7 [3]). By definition, $\mathfrak{P}_{\mathrm{rig}}$ is a rigid analytic space over $\mathrm{Sp}(K)$. As is well known (§7.1.10 [3]) there is a natural specialization map

$$\mathrm{sp} : \mathfrak{P}_{\mathrm{rig}} \to \mathfrak{P}.$$

For any subset $Z \subset \mathfrak{P}$, we define the *tube* of Z (in $\mathfrak{P}_{\mathrm{rig}}$) to be the pre-image $\mathrm{sp}^{-1}(Z) :=]Z[_{\mathfrak{P}}$. If $Z \subset \mathfrak{P}$ is closed then $]Z[_{\mathfrak{P}}$ naturally has the structure of a rigid space over K. For the diagonal embedding $Y \to \mathfrak{P} \times \mathfrak{P}$ we denote the tube of Y (in $\mathfrak{P}_{\mathrm{rig}} \times \mathfrak{P}_{\mathrm{rig}}$) by

$]Y[_{\mathfrak{P}\times\mathfrak{P}}$. There are two natural projections

$$\begin{tikzcd} &]Y[_{\mathfrak{P}\times\mathfrak{P}} \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\ ]Y[_{\mathfrak{P}} & &]Y[_{\mathfrak{P}} \end{tikzcd}$$

A convergent isocrystal on Y is a locally free sheaf E on $]Y[_{\mathfrak{P}}$ together with an isomorphism

$$\phi : \mathrm{pr}_1^*(E) \cong \mathrm{pr}_2^*(E),$$

which restricts to the identity on the diagonal $]Y[_{\mathfrak{P}}\subset]Y[_{\mathfrak{P}\times\mathfrak{P}}$ and satisfies the usual cocycle condition. We observe that a convergent isocrystal naturally gives rise to an integrable connection on $]Y[_{\mathfrak{P}}$. We denote the category of convergent isocrystals on Y (with coefficients on K) by $\mathrm{Isoc}(Y/K)$. It is independent (up to canonical equivalence) on the choice of $\mathfrak{P}$ and embedding $Y \to \mathfrak{P}$. We remark that in general, such a $\mathfrak{P}$, together with a suitable closed embedding of Y, may not exist. However, because Y is separated and of finite type over $\mathrm{Spec}(\mathbb{F})$, we may always carry out this construction locally. In this case we define $\mathrm{Isoc}(Y/K)$ be first working locally and glueing. We remark that if Y is affine and smooth then we may find $\mathfrak{P}$, a formally smooth formal R-scheme, whose Special fibre is isomorphic to Y.

Let Frob be a fixed power of the absolute Frobenius of k. We fix, once and for all a homomorphism $\sigma : K \to K$ extending the natural action of Frob of $W(k) \subset K$, and fixing a uniformizer of R. We remark that this is always possible after making a finite extension of R. The category of convergent isocrystals on Y/K is naturally functorial in Y/K (§2.3.7 [1]). Hence the pair (Frob, σ) gives rise to a functor $F_\sigma^* : \mathrm{Isoc}(Y/K) \to \mathrm{Isoc}(Y/K)$. An convergent F-isocrystal on Y with coefficients in K, is a convergent isocrystal $E \in \mathrm{Isoc}(Y/K)$ endowed with an isomorphism

$$\Phi : F_\sigma^*(E) \cong E.$$

We denote the category of convergent F-isocrystals on Y/K by F-$\mathrm{Isoc}(Y/K)$.

3. p-adic Geometric Langlands for GL_1

Let X be a smooth projective curve over $\mathrm{Spec}(\mathbb{F})$ of genus g. Let $\mathrm{Pic}(X)$ denote the Picard variety of X, the moduli space of isomorphism classes of line bundles on X. Recall that $\mathrm{Pic}(X)$ has connected components $\mathrm{Pic}_d(X)$ indexed by $d \in \mathbb{Z}$ corresponding to the degree of the line bundle. The degree zero component, $\mathrm{Pic}_0(X)$, is the Jacobian variety of X and is an Abelian variety of rank g.

As in the introduction, there is a natural morphism

$$\begin{array}{rcl} h : X \times \mathrm{Pic}(X) & \to & \mathrm{Pic}(X) \\ (x, \mathcal{L}) & \mapsto & \mathcal{O}_X(x) \otimes \mathcal{L} := \mathcal{L}(x). \end{array}$$

For $d \in \mathbb{Z}$, we may restrict h to $\mathrm{Pic}_d(X)$ to give the morphism $h_d : X \times \mathrm{Pic}_d(X) \to \mathrm{Pic}_{d+1}(X)$.

For $d \in \mathbb{N}$, let X^d denote the d-fold product of X with itself over $\mathrm{Spec}(\mathbb{F})$. Let Sym_d denote the finite symmetric group of rank d. There is a natural action of Sym_d on X^d and we let $X^{(d)} := X^d/\mathrm{Sym}_d$ denote the d-fold symmetric product of X with itself. Because X is a non-singular projective curve, $X^{(d)}$ is smooth over $\mathrm{Spec}(\mathbb{F})$. By construction there is a natural projection

$$\mathrm{sym}^d : X^d \to X^{(d)}.$$

For $\underline{x} = \{x_1, \cdots, x_n\} \in X^{(d)}$ we have the line bundle $\mathcal{O}_X(\underline{x}) := \bigotimes_{i=1}^{i=n} \mathcal{O}_X(x_i)$. This induces the Abel-Jacobi map (of degree d):

$$\begin{array}{rcl} \pi_d : X^{(d)} & \to & \mathrm{Pic}_d(X) \\ \underline{x} & \mapsto & \mathcal{O}_X(\underline{x}). \end{array}$$

By the Riemann-Roch theorem, if $d > 2g-2$ then π_d is a projective bundle with fibres $\pi_d^{-1}(\mathcal{L}) = \mathbb{P}(H^0(X, \mathcal{L}))$ projective spaces of rank $d-g$.

The curve X is smooth and projective and hence admits a smooth formal lift $\mathfrak{P}$. Let $E \in F\text{-Isoc}(X/K)$ be of rank n. By definition, E may be interpreted as a locally free sheaf of rank n on $\mathfrak{P}_{\mathrm{rig}}$, together with the structure of a convergent F-isocrystal. For $d \in \mathbb{N}$, let $\mathfrak{P}^d_{\mathrm{rig}}$ and $\mathfrak{P}^{(d)}_{\mathrm{rig}}$ denote the d-fold product of $\mathfrak{P}_{\mathrm{rig}}$ with itself and the corresponding symmetric product. Because X is smooth, one dimensional and integral, both of these rigid spaces are smooth. The exterior tensor product $E^{\boxtimes d}$ can be interpreted as a locally free sheaf on $\mathfrak{P}^d_{\mathrm{rig}}$ together with a natural convergent F-isocrystal structure. We again have a natural projection

$$\mathrm{sym}^d : \mathfrak{P}^d_{\mathrm{rig}} \to \mathfrak{P}^{(d)}_{\mathrm{rig}}.$$

Using this map we may pushforward the sheaf underlying $E^{\boxtimes d}$ to give a coherent sheaf on $\mathfrak{P}^{(d)}_{\mathrm{rig}}$. This sheaf is naturally equipped with an action of Sym_d and we may take invariants with respect to this action. We thus define the sheaf $E^{(d)} := \mathrm{sym}^d_*(E^{\boxtimes d})^{\mathrm{Sym}_d}$. From now on fix the assumption that E is of rank one.

Proposition 1. $E^{(d)}$ naturally has the structure of a convergent F-isocrystal on $X^{(d)}$, with coeffients in K.

Proof. The category of convergent isocrystals is Zariski local hence, as above, we may work over Y, a smooth affine curve over $\mathrm{Spec}(\mathbb{F})$. As above, let $\mathfrak{P}$ be a formally smooth R-formal scheme over whose special fibre is Y. Let $\mathfrak{P}_{\mathrm{rig}}$ denote the generic fibre of $\mathfrak{P}$. Let Y^d be the d-fold product of Y with itself over $\mathrm{Spec}(\mathbb{F})$ and let $Y^{(d)}$ be the symmetric quotient. Let $\mathfrak{P}^d$ be the d-fold product of $\mathfrak{P}$ with itself over $\mathrm{Sp}(K)$ and let $\mathfrak{P}^{(d)}$ be the symmetric quotient. Let $M \in F\text{-Isoc}(Y/K)$ of rank one. By definition, M is a locally free sheaf of rank one on $\mathfrak{P}_{\mathrm{rig}}$ equipped with a convergent F-isocrystal structure. The d-fold exterior tensor of M with itself, $M^{\boxtimes d}$ is a locally free sheaf on $\mathfrak{P}^d_{\mathrm{rig}}$ equipped with the natural isocrystal structure. We have the isomorphisms

$$]Y^d[_{\mathfrak{P}^d\times\mathfrak{P}^d} \cong]Y[^d_{\mathfrak{P}\times\mathfrak{P}}$$

and

$$]Y^{(d)}[_{\mathfrak{P}^{(d)}\times\mathfrak{P}^{(d)}} \cong]Y[^{(d)}_{\mathfrak{P}\times\mathfrak{P}}.$$

Hence we get the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{P}^d_{\mathrm{rig}} \arrow[d, "\alpha"] &]Y[^d_{\mathfrak{P}\times\mathfrak{P}} \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d] & \mathfrak{P}^d_{\mathrm{rig}} \arrow[d, "\alpha"] \\
\mathfrak{P}^{(d)}_{\mathrm{rig}} &]Y[^{(d)}_{\mathfrak{P}\times\mathfrak{P}} \arrow[l, "q_1"'] \arrow[r, "q_2"] & \mathfrak{P}^{(d)}_{\mathrm{rig}},
\end{tikzcd}$$

Where the horizontal arrows are the natural projections and all vertical arrows are symmetric quotient projections. The pushforward $\alpha_*(M^{\boxtimes d})$ is a coherent sheaf on $\mathfrak{P}^{(d)}_{\mathrm{rig}}$ which comes equipped with a natural action of Sym_d. Because M is locally free of rank one the subsheaf $\alpha_*(M^{\boxtimes d})^{\mathrm{Sym}_d} \subset \alpha_*(M^{\boxtimes d})$ is again locally free of rank one. We must show that this sheaf comes equipped with a convergent integrable connection. The convergent integrable connection on $M^{\boxtimes d}$ is given by an isomorphism

$$\phi : p_1^*(M^{\boxtimes d}) \cong p_2^*(M^{\boxtimes d}).$$

We may push this forward by the map β to give:

$$\beta_*(\phi) : \beta_*(p_1^*(M^{\boxtimes d})) \cong \beta_*(p_2^*(M^{\boxtimes d})).$$

This isomorphism commutes with the natural action of Sym_d on both sides, hence we get an isomorphism

$$\beta_*(\phi) : (\beta_*(p_1^*(M^{\boxtimes d})))^{\mathrm{Sym}_d} \cong (\beta_*(p_2^*(M^{\boxtimes d})))^{\mathrm{Sym}_d}.$$

There are canonical isomorphisms:

$$q_1^*(\alpha_*(M^{\boxtimes d})^{\mathrm{Sym}_d}) \cong (\beta_*(p_1^*(M^{\boxtimes d})))^{\mathrm{Sym}_d}, \;\; q_2^*(\alpha_*(M^{\boxtimes d})^{\mathrm{Sym}_d}) \cong (\beta_*(p_2^*(M^{\boxtimes d})))^{\mathrm{Sym}_d}.$$

Putting everything together gives an isomorphism:

$$\beta_*(\phi) : q_1^*(\alpha_*(M^{\boxtimes d})^{\mathrm{Sym}_d}) \cong q_2^*(\alpha_*(M^{\boxtimes d})^{\mathrm{Sym}_d}).$$

We leave it as an exercise to the reader to check that $\beta_*(\phi)$ satisfies the usual cocycle conditions and thus gives $\alpha_*(M^{\boxtimes d})^{\mathrm{Sym}_d}$ the structure of a convergent isocrystal.

We have shown that this construction gives rise to a functor $\mathcal{G} : \mathrm{Isoc}(X^d/K) \to \mathrm{Isoc}(X^{(d)}/K)$. It is clear that $\mathcal{G}$ commutes with the natural Frobenius functors on $\mathrm{Isoc}(X^d/K)$ and $\mathrm{Isoc}(X^{(d)}/K)$. Hence $M^{(d)}$ naturally comes equipped with a Frobenius structure. The result follows. □

Observe that for $d \in \mathbb{N}$ there is a unique morphism $\phi_d : X \times X^{(d)} \to X^{(d+1)}$ such that the following diagram commutes:

$$\begin{tikzcd} X \times X^{(d)} \arrow[r, "\phi_d"] \arrow[d, "id\times\pi_d"] & X^{(d+1)} \arrow[d, "\pi_{d+1}"'] \\ X \times \mathrm{Pic}_d(X) \arrow[r, "h_d"] & \mathrm{Pic}_{d+1}(X). \end{tikzcd}$$

By construction there is an isomorphism $E \boxtimes E^{(d)} \cong \phi_d^*(E^{(d+1)})$, for all $d \in \mathbb{N}$.

Proposition 2. For $d > 2g-2$, there is a convergent F-isocrystal $\mathrm{Aut}_d(E) \in F$-$\mathrm{Isoc}(\mathrm{Pic}_d(X)/K)$ such that $\pi_d^*(\mathrm{Aut}_d(E)) \cong E^{(d)}$.

Proof. By the Riemann-Roch theorem, if $d > 2g-2$ then π_d is a projective bundle with fibres $\pi_d^{-1}(\mathcal{L}) = \mathbb{P}(H^0(X, \mathcal{L}))$. By 4.13 of [5], the pullback functor $\pi_d^* : \mathrm{Isoc}(\mathrm{Pic}_d(X)/K) \to \mathrm{Isoc}(X^{(d)}/K)$ is an equivalence of categories. Because pullback commutes with Frobenius we deduce that this induces an equivalence of categories $\pi_d^* : F\text{-}\mathrm{Isoc}(\mathrm{Pic}_d(X)/K) \to F$-$\mathrm{Isoc}(X^{(d)}/K)$. Hence we may choose $\mathrm{Aut}_d(E) \in F\text{-}\mathrm{Isoc}(\mathrm{Pic}_d(X)/K)$ such that $\pi_d^*(\mathrm{Aut}_d(E)) \cong E^{(d)}$. □

By the above commutative diagram, if $d > 2g-2$ then there is an isomorphism:

$$E \boxtimes \mathrm{Aut}_d(E) \cong h_d^*(\mathrm{Aut}_{d+1}(E)).$$

We now have to extend this construction to $\mathrm{Pic}_d(X)$ for $d \leq 2g-2$. Let $\check{E} \in F$-$\mathrm{Isoc}(X/K)$ denote the dual isocrystal. For $d \in \mathbb{Z}$ let $p_{1,d}$ and $p_{2,d}$ denote the natural projections from $X \times \mathrm{Pic}_d(X)$ to X and $\mathrm{Pic}_d(X)$ respectively. As just mentioned, the key observation is that for $d > 2g-2$ we have an isomorphism

$$p_{1,d}^*(\check{E}) \otimes h_d^*(\mathrm{Aut}_{d+1}(E)) \cong p_{2,d}^*(\mathrm{Aut}_d(E)).$$

This suggests that we should extend the construction to all of $\mathrm{Pic}(X)$ inductively using this property.

Proposition 3. Let $d \in \mathbb{Z}$ and $M \in F\text{-}\mathrm{Isoc}(\mathrm{Pic}_{d+1}(X)/K)$ with the property that there exists $N \in F\text{-}\mathrm{Isoc}(\mathrm{Pic}_{d+2}(X)/K)$ such that $E \boxtimes M \cong h_{d+1}^*(N)$. Then there exists an isocrystal $L \in F\text{-}\mathrm{Isoc}(\mathrm{Pic}_d(X)/K)$ such that

$$E \boxtimes L \cong h_d^*(M).$$

PROOF. The first observation to make is that $p_{2,d} : X \times \mathrm{Pic}_d(X) \to \mathrm{Pic}_d(X)$ is both proper and surjective. In the terminology of §4 of [5] this means that it satisfies descent. If we can show that $p^*_{1,d}(\check{E}) \otimes h^*_d(M)$ comes canonically equipped with descent data then we know there there must exist $L \in F\text{-Isoc}(\mathrm{Pic}_d(X)/K)$ such that $p^*_{1,d}(\check{E}) \otimes h^*_d(M) \cong p^*_{2,d}(L)$. This L will satisfy the statement of the proposition. We must therefore show that $p^*_{1,d}(\check{E}) \otimes h^*_d(M)$ comes with canonical descent data.

First observe that $(X \times \mathrm{Pic}_d(X)) \times_{\mathrm{Pic}_d(X)} (X \times \mathrm{Pic}_d(X)) \cong X^2 \times \mathrm{Pic}_d(X)$. The two natural projections are given by

$$\begin{aligned} \mathrm{pr}_1 : X^2 \times \mathrm{Pic}_d(X) &\to X \times \mathrm{Pic}_d(X)) \\ (x_1, x_2, \mathcal{L}) &\mapsto (x_1, \mathcal{L}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{pr}_2 : X^2 \times \mathrm{Pic}_d(X) &\to X \times \mathrm{Pic}_d(X) \\ (x_1, x_2, \mathcal{L}) &\mapsto (x_2, \mathcal{L}). \end{aligned}$$

If $S \in F\text{-Isoc}(X \times \mathrm{Pic}_d(X))$ then descent data (for the natural projection to $\mathrm{Pic}_d(X)$) is an isomorphism

$$\phi : \mathrm{pr}^*_1(S) \cong \mathrm{pr}^*_2(S),$$

satisfying the usual cocycle conditions. Let $\varphi : X^2 \times \mathrm{Pic}_d(X) \to X^2$ denote the natural projection. There are two natural maps

$$\begin{aligned} q_1 : X^2 \times \mathrm{Pic}_d(X) &\to X \times \mathrm{Pic}_{d+1}(X) \\ (x_1, x_2, \mathcal{L}) &\mapsto (x_1, \mathcal{L}(x_2)) \end{aligned}$$

and

$$\begin{aligned} q_2 : X^2 \times \mathrm{Pic}_d(X) &\to X \times \mathrm{Pic}_{d+1}(X) \\ (x_1, x_2, \mathcal{L}) &\mapsto (x_2, \mathcal{L}(x_1)). \end{aligned}$$

By construction $h_{d+1} \circ q_1 = h_{d+1} \circ q_2$. Thus there is an isomorphism $q^*_1(h^*_{d+1}(N)) \cong q^*_2(h^*_{d+1}(N))$. This induces an isomorphism $q^*_1(E \boxtimes M) \cong q^*_2(E \boxtimes M)$. Finally we get an isomorphism

$$\varphi^*(\check{E}^{\boxtimes 2}) \otimes q^*_1(E \boxtimes M) \cong \varphi^*(\check{E}^{\boxtimes 2}) \otimes q^*_2(E \boxtimes M).$$

Note that we also have isomorphisms:

$$\varphi^*(\check{E}^{\boxtimes 2}) \otimes q^*_1(E \boxtimes M) \cong \mathrm{pr}^*_1(p^*_{1,d}(\check{E}) \otimes h^*_d(M)), \;\; \varphi^*(\check{E}^{\boxtimes 2}) \otimes q^*_2(E \boxtimes M) \cong \mathrm{pr}^*_2(p^*_{1,d}(\check{E}) \otimes h^*_d(M)).$$

Putting all this together gives an isomorphism

$$\delta : \mathrm{pr}^*_1(p^*_{1,d}(\check{E}) \otimes h^*_d(M)) \cong \mathrm{pr}^*_2(p^*_{1,d}(\check{E}) \otimes h^*_d(M)).$$

It is a straightforward exercise to check that this satisfies the necessary cocycle conditions. Thus $p^*_{1,d}(\check{E}) \otimes h^*_d(M)$ comes equipped with canonical descent data for the morphism $p_{2,d}$. Thus there exists $L \in F\text{-Isoc}(\mathrm{Pic}_d(X)/K)$ with the desired property. $\square$

Theorem 1. Let $E \in F\text{-Isoc}(X/K)$ of rank one. Then there exists $\mathrm{Aut}(E) \in F$-$\mathrm{Isoc}(\mathrm{Pic}(X)/K)$ such that $E \boxtimes \mathrm{Aut}(E) \cong h^*(\mathrm{Aut}(E))$.

Proof. To construct $\mathrm{Aut}(E)$ it is sufficient to construct it on $\mathrm{Pic}_d(X)$ for each $d \in \mathbb{Z}$. If $d > 2g-2$ we take $\mathrm{Aut}_d(E)$ as defined above. Using the previous proposition we inductively extend to give an isocrystal $\mathrm{Aut}_d(E) \in F\text{-Isoc}(\mathrm{Pic}_d(X)/K)$ for all $d \leq 2g-2$. We define $\mathrm{Aut}(E) = \coprod_{d\in\mathbb{Z}} \mathrm{Aut}_d(X)$. By construction $\mathrm{Aut}(E)$ satisfies the property $E \boxtimes \mathrm{Aut}(E) \cong h^*(\mathrm{Aut}(E))$. $\square$

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