Principal, Prime and Maximal Ideals

R — commutative ring

Definition An ideal $I \subset R$ is proper if $I \neq R$.

Note that $I \subset R$ proper $\iff$ R/I non-trivial ring

Definition An ideal $I \subset R$ is principal $\iff \exists\, a \in I$ such that $I = \{ra \mid r \in R\}$. In this case we write $I = (a)$.

Remark: Given any $a \in R$, $(a) \subset R$ is always an ideal.

Example 1/ $m\mathbb{Z} \subset \mathbb{Z}$ is a principal ideal with $m\mathbb{Z} = (m)$.

2/ Non-example: $I = \{f(x,y) \in \mathbb{Q}[x,y] \mid f(0,0) = 0\}$

$f(x,y) \in I \iff f(x,y)$ has 0 constant term

e.g. $x, y, x+y, xy^2 + 2x^3 \in I$

$\Rightarrow$ No non-zero constant polynomials are in I

Both $x, y \in I$, however the only polynomials dividing both are constant. Hence I is not principal.

Definition Let $I \subset R$ be an ideal. We say I is prime if

1/ I is proper

2/ $ab \in I \Rightarrow a \in I$ or $b \in I$ $\forall a, b \in R$

<u>Example</u> $m\mathbb{Z} \subset \mathbb{Z}$ is a prime ideal $\iff$ m prime
$(m \in \mathbb{N})$

<u>Proposition</u> $I \subset R$ prime $\iff$ R/I an integral domain

<u>Proof</u> $(\Rightarrow)$

1. $I \subset R$ prime $\Rightarrow$ I proper $\Rightarrow$ R/I non-trivial
2. R commutative $\Rightarrow$ R/I commutative
3. Let $a+I, b+I \subset R/I$ and assume

$$(a+I)(b+I) = 0_R + I = 0_{R/I}$$

$$\Rightarrow \quad ab + I = 0_R + I$$

$$\Rightarrow \quad ab \in I \quad \underset{I \text{ prime}}{\Rightarrow} \quad a \in I \text{ or } b \in I$$

$$a \in I \Rightarrow a + I = 0_R + I = 0_{R/I}$$

$$b \in I \Rightarrow b + I = 0_R + I = 0_{R/I}$$

$\Rightarrow$ R/I an integral domain

$(\Leftarrow)$

1. R/I integral domain $\Rightarrow$ R/I non-trivial $\Rightarrow$ I proper
2. Let $a, b \in R$ s.t. $ab \in I$ $\Rightarrow$

$$ab + I = 0_R + I = 0_{R/I}$$

$$\Rightarrow (a+I)(b+I) = 0_{R/I}$$

$$\underset{R/I \text{ integral domain}}{\Rightarrow} \quad a + I = 0 + I \quad \text{or} \quad b + I = 0 + I$$

$$\Rightarrow \quad a \in I \text{ or } b \in I$$

$\Rightarrow$ $I \subset R$ is prime

$\square$

<u>Definition</u> Let $I \subset R$ be an ideal. We say I is <u>maximal</u> if

1/ I proper

2/ If $J \subset R$ is an <u>ideal</u> s.t. $I \subset J$ then $J = I$ or $J = R$.

There are no strictly intermediate ideals between I and R

<u>Proposition</u> Let $I \subset R$ be an ideal. Then I maximal $\Leftrightarrow$ R/I a field

<u>Proof</u> $(\Rightarrow)$

1/ I maximal $\Rightarrow$ I proper $\Rightarrow$ R/I non-trivial

2/ R commutative $\Rightarrow$ R/I commutative

3/ Let $a + I \in R/I \setminus \{0_R + I\}$

We must show $a + I$ has a multiplicative inverse.

$a + I \neq 0_R + I \Rightarrow a \notin I$

Let $(a) + I := \{ra + b \mid r \in R, b \in I\}$

<u>Claim</u> $(a) + I \subset R$ is an ideal

1/ $0_R \in I \Rightarrow 0_R \cdot a + 0_R = 0_R \in (a) + I$

2/ $b \in I \Rightarrow -b \in I$. Hence $ra + b \in (a) + I$

$\Rightarrow (-r)a + (-b) \in (a) + I$

3/ $(r_1 a + b_1) + (r_2 a + b_2) = (r_1 + r_2)a + (b_1 + b_2)$

$b_1, b_2 \in I \Rightarrow b_1 + b_2 \in I$

$\Rightarrow (r_1 a + b_1) + (r_2 a + b_2) \in I$

4/ If $b \in I$ and $s \in R \Rightarrow sb \in I$

$\Rightarrow s(ra + b) = (sr)a + sb \in (a) + I$ $\square$

Let $J = (a) + I$. $I \subset J$ and $a \in J$,

$a \notin I \Rightarrow I \neq J. \underset{I \text{ maximal}}{\Rightarrow} J = R$

$\Rightarrow 1_R \in J$

$\Rightarrow \exists\, r \in R$ and $b \in I$ s.t. $ra + b = 1_R$

$\Rightarrow (ra) + I = 1_R + I = 1_{R/I}$

$\Rightarrow (r + I)(a + I) = (a + I)(r + I) = 1_{R/I}$

$\Rightarrow a + I$ has a multiplicative inverse.

$\Rightarrow R/I$ a field.

($\Leftarrow$) Assume R/I a field.

1) R/I a field $\Rightarrow R/I$ non-trivial $\Rightarrow I$ proper

2) Assume I is not maximal. Then there is an ideal $J \subset R$ s.t. $I \subsetneq J \subsetneq R$

$J \neq I \Rightarrow \exists\, a \in J$ s.t. $a \notin I \Rightarrow a + I \neq 0_{R/I}$

$J \neq R \Rightarrow 1_R \notin J$

R/I a field $\Rightarrow \exists\, r + I \in R/I$ s.t.

$(a + I)(r + I) = (r + I)(a + I) = 1_R + I$

$= ra + I = 1_R + I$

$\Rightarrow \exists\, b \in I$ s.t. $ra + b = 1_R$

$$\left.\begin{array}{l} b \in I \Rightarrow b \in J \\ a \in J \Rightarrow ra \in J \end{array}\right\} \Rightarrow ra + b \in J \Rightarrow 1_R \in J$$

$\Rightarrow J = R$

Contradiction. Therefore $I \subset R$ is maximal. $\square$

Corollary Let $I \subset R$ be an ideal. Then

$$I \text{ maximal} \Rightarrow I \text{ prime}$$

Proof I maximal $\Rightarrow R/I$ a field $\Rightarrow R/I$ integral domain

$\Rightarrow I$ prime

$\square$