Normal Subgroups and Isomorphism Theorems

Let $(G,*)$ and $(H,\circ)$ be groups and $\phi : G \longrightarrow H$ be a homomorphism.

Definition Kernel of ϕ

$$Ker(\phi) := \{ g \in G \mid \phi(g) = e_H \} \subset G$$

Remark

m×n matrix — Null space of A

If $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\underline{x} \longmapsto A\underline{x}$, then $Ker(\phi) = Nul(A)$

Proposition $Ker(\phi) \subset G$ is a subgroup.

Proof

- $\phi(e_G) = e_H \Rightarrow e_G \in Ker(\phi)$
- $x, y \in Ker(\phi) \Rightarrow \phi(x), \phi(y) = e_H \Rightarrow \phi(x) \circ \phi(y) = e_H \circ e_H = e_H$
  $\Rightarrow \phi(x * y) = e_H \Rightarrow x * y \in Ker(\phi)$
- $x \in Ker(\phi) \Rightarrow \phi(x) = e_H \Rightarrow (\phi(x))^{-1} = e_H^{-1} = e_H$
  $\Rightarrow \phi(x^{-1}) = e_H \Rightarrow x^{-1} \in Ker(\phi)$

$\square$

Proposition ϕ injective $\Leftrightarrow Ker(\phi) = \{e_G\}$

Proof

$(\Rightarrow)$ $\phi(e_G) = e_H$ Hence ϕ injective $\Rightarrow Ker(\phi) = \{e_G\}$

$(\Leftarrow)$ Assume $Ker(\phi) = \{e_G\}$ and $\phi(x) = \phi(y)$ for $x, y \in G$

$\Rightarrow \phi(x) \circ (\phi(y))^{-1} = e_H \Rightarrow \phi(x) \circ \phi(y^{-1}) = e_H \Rightarrow \phi(x * y^{-1}) = e_H$

$\Rightarrow x * y^{-1} \in Ker\,\phi \Rightarrow x * y^{-1} = e_G \Rightarrow x = y$

$\square$

<u>Observation</u> : If $x \in Ker(\phi)$ and $g \in G$

$\Rightarrow \quad \phi(g * x * g^{-1}) = \phi(g) \circ \phi(x) \circ \phi(g^{-1}) = \phi(g) \circ \phi(g)^{-1} = e_H$

$\Rightarrow \quad g * x * g^{-1} \in Ker(\phi)$

<u>Conclusion</u> : $Ker(\phi) \subset G$ is closed under conjugation by all of G.

<u>Definition</u> Let $(G, *)$ be a group and $N \subset G$ a subgroup. We say N is <u>normal</u> in G if

Does not need to equal x

$x \in N, g \in G \Rightarrow g * x * g^{-1} \in N$

normal subgroup

We write $N \triangleleft G$.

<u>Remarks</u>

- $Ker(\phi) \triangleleft G$
- $N \triangleleft G \Leftrightarrow N$ union of conjugacy classes

For example $\{\underbrace{e}_{\substack{\text{All cycles} \\ \text{with cycle structure} \\ \{1,1,1\}}}, \underbrace{(123), (132)}_{\substack{\text{All cycles with} \\ \text{cycle structure } \{3\}}}\} = gp(\{(123)\}) \triangleleft Sym_3$

However $\{e, (12)\} = gp(\{(12)\}) \ntriangleleft Sym_3$

(13) has same cycle structure so is a conjugate

- G Abelian $\Rightarrow$ Every subgroup is normal

<u>Definition</u> We say G is <u>simple</u> if $N \triangleleft G \Rightarrow N = \{e_G\}$ or $N = G$

Examples $\mathbb{Z}/p\mathbb{Z}$ (prime), Alt_n $n \geq 5$ (Not obvious)

Theorem Let $N \trianglelefteq G$ then the binary operation

$$G/N \times G/N \longrightarrow G/N$$

$$(xN, yN) \longmapsto x*y\,N$$

is well-defined and gives G/N the structure of a group. (Quotient group)

Proof (Outline)

Let $x_1, x_2, y_1, y_2 \in G$ and $x_1 N = x_2 N$, $y_1 N = y_2 N$

$\Leftrightarrow$ $x_1^{-1} * x_2 \in N$, $y_1^{-1} * y_2 \in N$

We want to show this implies $x_1 * y_1 N = x_2 * y_2 N$

$$(x_1 * y_1)^{-1} * (x_2 * y_2) = y_1^{-1} * x_1^{-1} * x_2 * y_2 = \underbrace{\underbrace{y_1^{-1} * \overbrace{(x_1^{-1} * x_2)}^{\in N} * y_1}_{\in N} * \overbrace{y_1^{-1} * y_2}^{\in N}}_{\in N}$$

$\Rightarrow (x_1 * y_1)^{-1} * (x_2 * y_2) \in N \Rightarrow x_1 * y_1 N = x_2 * y_2 N$

$\Rightarrow$ Binary Operation is well-defined

- $\forall\, xN, yN, zN \in G/N$

  $(xN * yN) * zN = (x*y)*z\,N = x*(y*z)N = xN * (yN * zN)$
- $\forall\, xN \in G/N$, $xN * eN = eN * xN = xN$
- Given $xN \in G/N$, $xN * x^{-1}N = x^{-1}N * xN = eN$

$\square$

Example $G = \mathbb{Z}$, $N = m\mathbb{Z}$ $\quad G/N = \mathbb{Z}/m\mathbb{Z}$.

Remark

- The map $\phi : G \longrightarrow G/N$ is called the quotient homomorphism
$$x \longmapsto xN$$

$\ker(\phi) = N$.

The First Isomorphism Theorem

Let $\phi : G \longrightarrow H$ be a homomorphism. Then the map
$$\psi : G/\mathrm{Ker}(\phi) \longrightarrow \mathrm{Im}(\phi)$$
$$x\,\mathrm{Ker}(\phi) \longrightarrow \phi(x)$$
is a well-defined isomorphism

Proof

- $x\,\mathrm{Ker}(\phi) = y\,\mathrm{Ker}(\phi) \Leftrightarrow x^{-1} * y \in \mathrm{Ker}(\phi) \Leftrightarrow \phi(x^{-1} * y) = e_H$

$\Leftrightarrow (\phi(x))^{-1} \circ \phi(y) = e_H \Leftrightarrow \phi(x) = \phi(y)$

This proves ψ is well-defined and injective.

- ψ surjective by definition of $\mathrm{Im}(\phi)$
- $\psi\big((x\,\mathrm{Ker}(\phi)) * (y\,\mathrm{Ker}(\phi))\big) = \psi(x*y\,\mathrm{Ker}(\phi)) = \phi(x*y)$

$= \phi(x) \circ \phi(y) = \psi(x\,\mathrm{Ker}(\phi)) \circ \psi(y\,\mathrm{Ker}(\phi))$ $\square$

Corollary $|G| < \infty \Rightarrow |G| = |\mathrm{Ker}(\phi)| \cdot |\mathrm{Im}(\phi)|$

Proof $G/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi) \Rightarrow |G/\mathrm{Ker}(\phi)| = |\mathrm{Im}(\phi)|$

$\Rightarrow \dfrac{|G|}{|\mathrm{Ker}\,\phi|} = |\mathrm{Im}(\phi)| \Rightarrow |G| = |\mathrm{Ker}(\phi)| \cdot |\mathrm{Im}(\phi)|$

$\square$

Third Isomorphism Theorem

Let $N \lhd G$. Then there is a natural bijection

$$\left\{ \begin{matrix} \text{Subgroups of} \\ G \text{ containing } N \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Subgroups of} \\ G/N \end{matrix} \right\}$$

$$H \longmapsto H/N = \{ xN \mid x \in H \}$$

Moreover, $H/N \lhd G/N \iff H \lhd G$ and in this case

$$G/H \cong \frac{G/N}{H/N}$$

<u>Proof</u> : See Notes.