Limits

Motivating Example: A company is making a product. After analysing their costs they decide the cost function is

$$C(x) = \frac{x-4}{\sqrt{x}-2}$$

number of units made (in millions) — x

cost of making x units (in millions) — $C(x)$

Q_1: What happens to $C(x)$ as x gets closer and closer to (approaches) 4?

Problem: $C(x)$ is not defined when $x = 4$.

1/ (Table)

x approaching 4 from below →

← x approaching 4 from above

x	3	3.5	3.9	4	4.1	4.5	5
$C(x)$	3.73	3.87	3.97	↑ undefined	4.02	4.12	4.24

$C(x)$ approaches 4 →

← $C(x)$ approaches 4

2/ (Algebra)

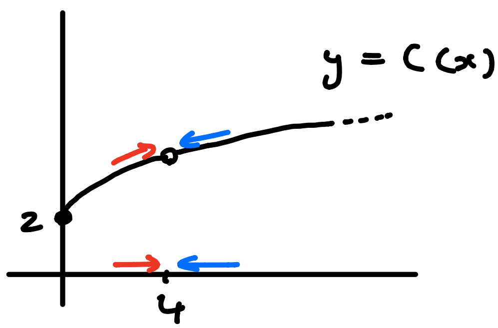
$$C(x) = \frac{x-4}{\sqrt{x}-2} = \frac{(\sqrt{x}+2)(\sqrt{x}-2)}{\sqrt{x}-2}$$

$$\Rightarrow C(x) = \begin{cases} \sqrt{x}+2 & \text{if } x \neq 4 \\ \text{undefined} & \text{if } x = 4 \end{cases}$$

As x approaches 4 from above or below $\sqrt{x}+2$ approaches $\sqrt{4}+2 = 4$

3/ (Graph) $C(x) = \begin{cases} \sqrt{x} + 2 & \text{if } x \neq 4 \\ \text{undefined} & \text{if } x = 4 \end{cases}$

Picture



Conclusion : As x approaches 4 from either above or below, $C(x)$ approaches 4.

Definition Let f be a function and a, L be two real numbers.

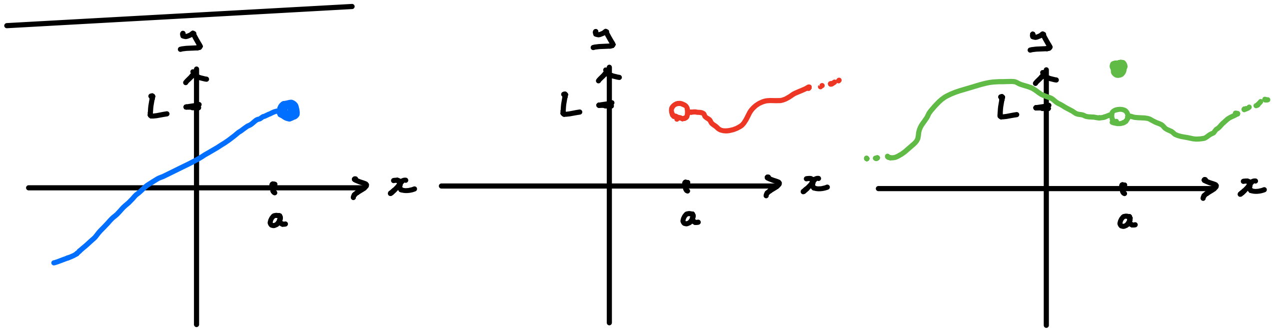
$\lim\limits_{x \to a^+} f(x) = L \iff$ $f(x)$ approaches L as x approaches a from strictly above.

$\lim\limits_{x \to a^-} f(x) = L \iff$ $f(x)$ approaches L as x approaches a from strictly below.

$\lim\limits_{x \to a} f(x) = L \iff$ $f(x)$ approaches L as x approaches (but does not equal) a from above and below.

L = Limit of f as x approaches a.
(from above / below / both sides)

Basic Graphs :



$\lim_{x \to a^-} f(x) = L$ $\qquad$ $\lim_{x \to a^+} f(x) = L$ $\qquad$ $\lim_{x \to a} f(x) = L$

Remarks: 1/ In first example $\lim_{x \to 4^+} C(x) = 4$, $\lim_{x \to 4^-} C(x) = 4$ and $\lim_{x \to 4} C(x) = 4$

2/ If no such L exists we say "limit does not exist" (DNE)

Calculating Limits

$$f(x) = \left.\begin{array}{l} \text{polynomial} \\ \text{rational function} \\ \text{power function} \\ \text{exponential function} \\ \text{Log function} \end{array}\right\} \Rightarrow \lim_{x \to a} f(x) = f(a)$$

(assuming $x = a$ in domain)

Example $\quad \lim_{x \to 5} x^2 + 7x - 2 = 5^2 + 7 \cdot 5 - 2 = 58$

Laws of Limits : Assume $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} g(x) = B$

1/ $\lim_{x \to a} (f(x) + g(x)) = A + B$

2/ $\lim_{x \to a} (k f(x)) = k A \quad$ (k constant)

3/ $\lim_{x \to a} (f(x) g(x)) = A B$

4/ $$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \begin{cases} \frac{A}{B} & \text{if } B \neq 0 \\ \text{DNE} & \text{if } A \neq 0,\ B = 0 \\ \text{UNCLEAR} & \text{if } A = 0,\ B = 0 \end{cases}$$

5/ $$\lim_{x\to a} (f(x))^r = A^r \quad (\text{If limit exists})$$

6/ $$\lim_{x\to a} b^{f(x)} = b^A$$

7/ $$\lim_{x\to a} \log_b(f(x)) = \log_b(A) \quad (\text{If limit exists})$$

Examples

1/ $$\lim_{x\to 1} \frac{x^2+1}{x^3+3} = \frac{1^2+1}{1^3+3} = \frac{2}{4} = \frac{1}{2}$$

2/ $$\lim_{x\to -2} \ln\left(\frac{x+2}{4-x^2}\right) = ?$$

$$\left.\begin{aligned} &\lim_{x\to -2} x+2 = -2+2 = 0 \\ &\lim_{x\to -2} 4-x^2 = 4-(-2)^2 = 0 \end{aligned}\right\} \text{Limit of quotient unclear.}$$

$$\frac{x+2}{4-x^2} = \frac{x+2}{(x+2)(2-x)} \underset{x\neq -2}{=} \frac{1}{2-x},$$

$$\Rightarrow \lim_{x\to -2} \frac{x+2}{4-x^2} = \lim_{x\to -2} \frac{1}{2-x} = \frac{1}{2-(-2)} = \frac{1}{4}$$

$$\Rightarrow \lim_{x\to -2} \ln\left(\frac{x+2}{4-x^2}\right) = \ln\left(\frac{1}{4}\right).$$

If we are an unclear quotient try simplifying using algebra.