

Abstract Algebra

Basics : Lectures MTWT 1015 Evans Hall 2pm - 4pm
Office Hours MTWTF 796 Evans Hall 12pm - 1pm
Homework due every Thursday in class (Not 1st week)
Midterm 7/19 in class
Final 8/9 in class
Prerequisites : Linear Algebra (S4 or equivalent)

What is Algebra?

Two objects combining to form a third.
e.g. addition of numbers

Algebra = Abstract study of composition.

The foundations of the subject are arithmetic.

Unity, The number 1



$\mathbb{N} = \{1, 2, 3, \dots\}$ the natural numbers.
Comes with + and \times .



$\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$ the integers.
Comes with + and \times .



$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \text{ integers}, b \neq 0 \right\}$ the rational numbers.
Comes with + and \times

As we go down
+ and \times
gain more
properties.
e.g. Given a in \mathbb{Z}
there exists b in \mathbb{Z}
such that
 $a + b = b + a = 0$

Exercise : Think back through past mathematics courses. Give as many examples of sets with some kind of composition as you can. Can you spot any recurring properties?

Linear Algebra gives important examples:

$(M_n(\mathbb{R}), +, \times)$ - $n \times n$ matrices with real entries together with matrix addition and multiplication.

$(GL_n(\mathbb{R}), \times)$ - $n \times n$ invertible matrices with real entries together with matrix multiplication

- | <u>$(\mathbb{Z}, +)$</u> | <u>$(GL_n(\mathbb{R}), \times)$</u> |
|--|--|
| • $(a+b)+c = a+(b+c)$
for all a, b, c in \mathbb{Z} . | • $(AB)C = A(BC)$
for all A, B, C in $GL_n(\mathbb{R})$. |
| • $a+0 = 0+a = a$
for all a in \mathbb{Z} . | • $A I_n = I_n A = A$
for all A in $GL_n(\mathbb{R})$. |
| • Given a in \mathbb{Z} , there exists b in \mathbb{Z} such that $a+b = b+a = 0$ | • Given A in $GL_n(\mathbb{R})$, there exists B in $GL_n(\mathbb{R})$ such that $AB = BA = I_n$ |

Shared Properties

Warning: There are still major differences. For example $a+b = b+a$ for all a, b in \mathbb{Z} , whereas there exist A, B in $GL_n(\mathbb{R})$ such that $AB \neq BA$.

- | <u>$(\mathbb{Z}, +, \times)$</u> | <u>$(M_n(\mathbb{R}), +, \times)$</u> |
|--|--|
| • $a+b = b+a$
for all a, b in \mathbb{Z} | • $A+B = B+A$
for all A, B in $M_n(\mathbb{R})$ |
| • $(ab)c = a(bc)$
for all a, b, c in \mathbb{Z} | • $(AB)C = A(BC)$
for all A, B, C in $GL_n(\mathbb{R})$. |
| • $a1 = 1a = a$
for all a in \mathbb{Z} | • $A I_n = I_n A = A$
for all A in $GL_n(\mathbb{R})$. |
| • Given a, b, c in \mathbb{Z}
$a(b+c) = ab+ac$ | • Given A, B, C in $M_n(\mathbb{R})$
$A(B+C) = AB+AC$ |

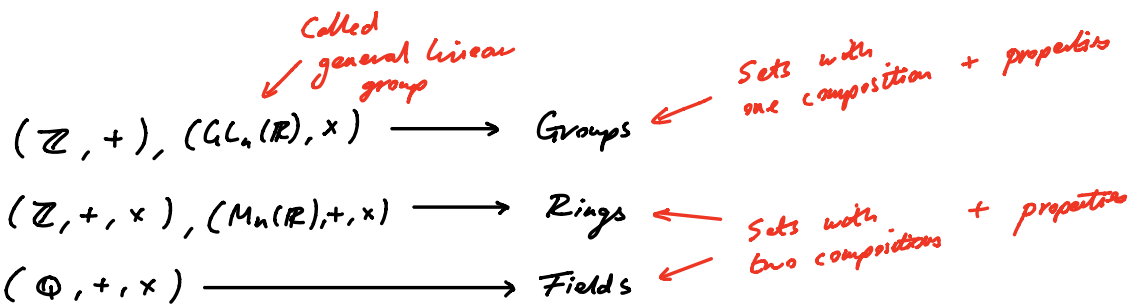
Notice $(\mathbb{Q}, +, \times)$ has extra property:

Given $a \neq 0$ in \mathbb{Q} , there exists b in \mathbb{Q} such that

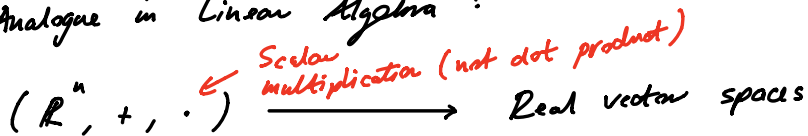
$$ab = ba = 1$$

Central Idea
in
Abstract Algebra

Define and study a broad class of objects
(sets with compositions) of which
 $(\mathbb{Z}, +)$, $(GL_n(\mathbb{R}), \times)$, $(\mathbb{Z}, +, \times)$, $(M_n(\mathbb{R}), +, \times)$ and
 $(\mathbb{Q}, +, \times)$ are definitive members



Analogue in Linear Algebra:



Remark

1/ The power of the subject comes from the fact groups, rings and fields permeate all mathematical sciences.

2/ Don't be fooled into thinking groups, rings and fields will always look like the above examples. We'll encounter many much more exotic examples throughout the course.