Abstract Algebra

Basics: Lectures MTWT 1015 Evans Hall 2pm – 4pm

Office Hours MTWTF 796 Evans Hall 12pm – 1pm

Homework due every Thursday in class (Not 1st week)

Midterm 7/19 in class

Final 8/9 in class

Prerequisites : Linear Algebra (54 or equivalent)

What is Algebra?

Algebra = Abstract Study of composition.

Two objects combining to form a third.
e.g. addition of numbers

The foundations of the subject are arithmetic.

Unity, The number 1

$\downarrow$

$\mathbb{N} = \{1, 2, 3, \ldots\}$ the natural numbers.
Comes with $+$ and $\times$.

$\downarrow$

$\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ the integers.
Comes with $+$ and $\times$.

$\downarrow$

$\mathbb{Q} = \{ \frac{a}{b} \mid a, b \text{ integers}, b \neq 0 \}$ the rational numbers.

Comes with $+$ and $\times$

As we go down $+$ and $\times$ gain more properties.
e.g. Given a in $\mathbb{Z}$ there exists b in $\mathbb{Z}$ such that
$a+b = b+a = 0$

Exercise: Think back through past mathematics courses. Give as many examples of sets with some kind of composition as you can. Can you spot any recurring properties?

Linear Algebra gives important examples :

$(M_n(\mathbb{R}), +, \times)$ - $n \times n$ matrices with real entries together with matrix addition and multiplication.

$(GL_n(\mathbb{R}), \times)$ - $n \times n$ invertible matrices with real entries together with matrix multiplication

$(\mathbb{Z}, +)$	$(GL_n(\mathbb{R}), \times)$
• $(a+b)+c = a+(b+c)$ for all a,b,c in $\mathbb{Z}$.	$(AB)C = A(BC)$ for all A,B,C in $GL_n(\mathbb{R})$.
• $a+0 = 0+a = a$ for all a in $\mathbb{Z}$.	$AI_n = I_nA = A$ for all A in $GL_n(\mathbb{R})$.
• Given a in $\mathbb{Z}$, there exists b in $\mathbb{Z}$ such that $a+b = b+a = 0$	Given A in $GL_n(\mathbb{R})$, there exists B in $GL_n(\mathbb{R})$ such that $AB = BA = I_n$

Shared Properties

Warning : There are still major differences. For example $a+b = b+a$ for all a,b in $\mathbb{Z}$, whereas there exist A,B in $GL_n(\mathbb{R})$ such that $AB \neq BA$.

$(\mathbb{Z}, +, \times)$	$(M_n(\mathbb{R}), +, \times)$
• $a+b = b+a$ for all a,b in $\mathbb{Z}$	$A+B = B+A$ for all A,B in $M_n(\mathbb{R})$
• $(ab)c = a(bc)$ for all a,b,c in $\mathbb{Z}$	$(AB)C = A(BC)$ for all A,B,C in $GL_n(\mathbb{R})$.
• $a1 = 1a = a$ for all a in $\mathbb{Z}$	$AI_n = I_nA = A$ for all A in $GL_n(\mathbb{R})$.
• Given a,b,c in $\mathbb{Z}$ $a(b+c) = ab+ac$	Given A,B,C in $M_n(\mathbb{R})$ $A(B+C) = AB+AC$

Notice $(\mathbb{Q}, +, \times)$ has extra property :

Given $a \neq 0$ in $\mathbb{Q}$, there exists b in $\mathbb{Q}$ such that
$ab = ba = 1$

Central Idea in Abstract Algebra : Define and study a broad class of objects (sets with compositions) of which $(\mathbb{Z}, +)$, $(GL_n(\mathbb{R}), \times)$, $(\mathbb{Z}, +, \times)$, $(M_n(\mathbb{R}), +, \times)$ and $(\mathbb{Q}, +, \times)$ are definitive members

Called general linear group

$(\mathbb{Z}, +)$, $(GL_n(\mathbb{R}), \times) \longrightarrow$ Groups $\leftarrow$ Sets with one composition + properties

$(\mathbb{Z}, +, \times)$, $(M_n(\mathbb{R}), +, \times) \longrightarrow$ Rings $\leftarrow$ Sets with two compositions + properties

$(\mathbb{Q}, +, \times) \longrightarrow$ Fields $\leftarrow$

Analogue in Linear Algebra :

Scalar multiplication (not dot product)

$(\mathbb{R}^n, +, \cdot) \longrightarrow$ Real vector spaces

Remark

1/ The power of the subject comes from the fact groups, rings and fields permeate all mathematical sciences.

2/ Don't be fooled into thinking groups, rings and fields will always look like the above examples. We'll encounter many much more exotic examples throughout the course.