Continuous Functions

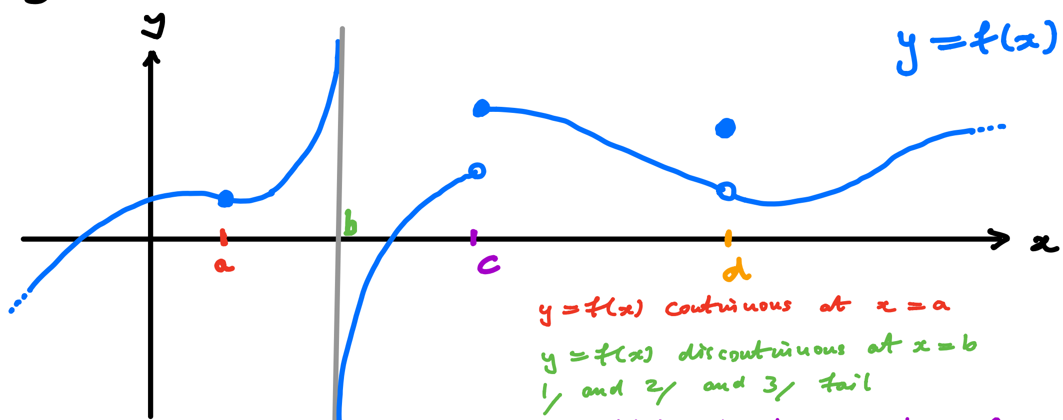
Definition A function f is continuous at $x = a$ if

1/ $f(a)$ is defined, i.e. a is in domain of f

2/ $\lim_{x \to a} f(x)$ exists

3/ $\lim_{x \to a} f(x) = f(a)$

If any of these three conditions fail we say f is discontinuous at $x = a$



$y = f(x)$ continuous at $x = a$

$y = f(x)$ discontinuous at $x = b$
1/ and 2/ and 3/ fail

$y = f(x)$ discontinuous at $x = c$
2/ and 3/ fail

$y = f(x)$ discontinuous at $x = d$
3/ fails

Intuition : $y = f(x)$ is continuous at $x = a$ if we can draw graph through $(a, f(a))$ without lifting pen.

Remarks

1/ In the definition we could replace $\lim_{x \to a} f(x)$

$\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$.

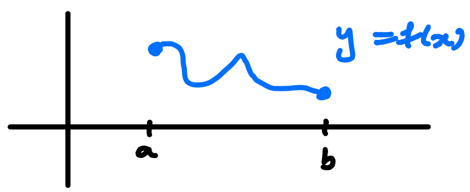
$\lim_{x \to a^+} f(x) = f(a) \iff f$ is <u>continuous from right</u> at $x = a$

$\lim_{x \to a^-} f(x) = f(a) \iff f$ is <u>continuous from left</u> at $x = a$

In above picture f is continuous from right but not the left at $x = b$.

2/ A function f is said to be <u>continuous</u> if it is continuous at every $x = a$ in its domain. (If $x = a$ is at endpoint of domain we demand it is continuous from right / left)

<u>Example</u> domain $= [a, b]$ (closed interval)



$\Rightarrow f$ continuous

Polynomials, Rational functions, exponential functions, Logarithmic functions and power functions are continuous

The sum / difference / product / quotient / composition of these functions is again continuous (although the domain may change)

Example $f(x) = \begin{cases} e^{(x^2-x)} & \text{if } x \leq 0 \\ \ln(x^2+1)+1 & \text{if } x > 0 \end{cases}$

Is $f(x)$ continuous at $x = 0$?

1/ $f(0) = e^{(0^2-0)} = e^0 = 1$

2/ $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \ln(x^2+1)+1 = \ln(0^2+1)+1$
$= 0+1 = 1$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} e^{x^2-x} = e^{0^2-0} = e^0 = 1$

$\Rightarrow \lim\limits_{x \to 0} f(x) = 1$

3/ $\lim\limits_{x \to 0} f(x) = 1 = f(0)$ $\Rightarrow$ f continuous at $x = 0$.