Higher Derivatives, Concavity and the Second Derivative Test

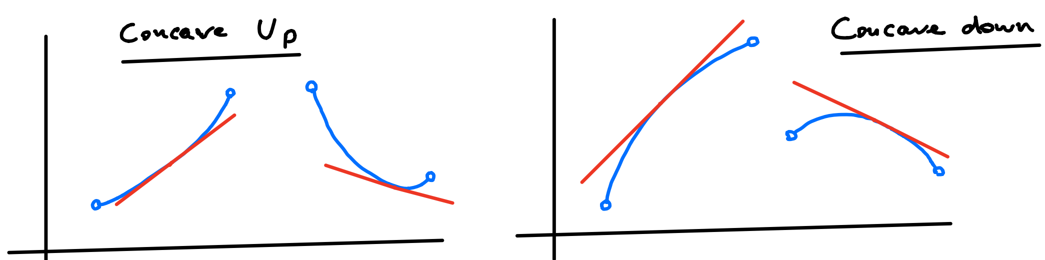
Aim: Understand when the graph of a function is "curved".

Definition

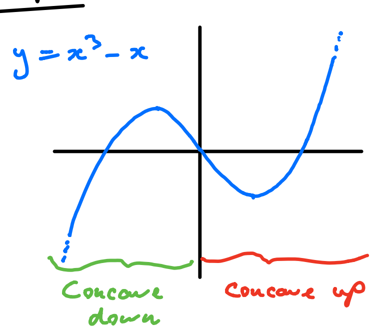
f is concave up on an interval (a, b) if the graph is above every tangent line.

f is concave down on an interval (a, b) if the graph is below every tangent line.

Basic Picture:



Example $f(x) = x^3 - x = x(x^2 - 1) = x(x+1)(x-1)$



$\Rightarrow$ f concave up on $(0, \infty)$

f concave down on $(-\infty, 0)$

Definition

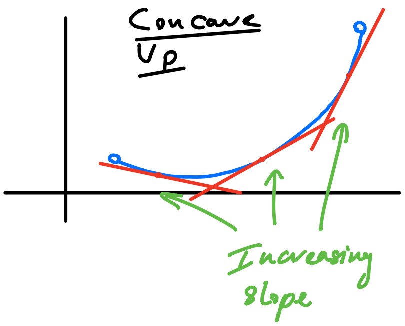
An inflection point of $(c, f(c))$ on the graph where

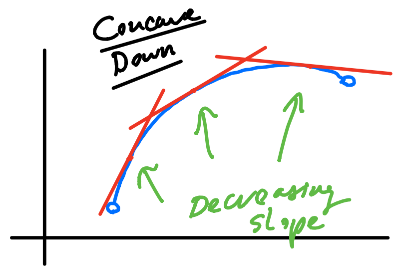
1/ f differentiable at c (ie $f'(c)$ exists)

2/ The graph changes concavity at $(c, f(c))$.

In the example $f(x) = x^3 - x$, $(0,0)$ is an inflection point.

Q: How can we determine the concavity of a function?





f' increasing on (a,b) $\Rightarrow$ f concave up on (a,b)

f' decreasing on (a,b) $\Rightarrow$ f concave down on (a,b)

Definition (Higher Derivatives)

$f''(x)$ = Derivative of $f'(x)$ = 2^{nd} Derivative of f

$f'''(x)$ = Derivative of $f''(x)$ = 3^{rd} Derivative of f

$\vdots$

$f^{(n)}(x)$ = Derivative of $f^{(n-1)}(x)$ = n^{th} Derivative of f

Alternate Notation : $f'(x) = \dfrac{df}{dx}$, $f''(x) = \dfrac{d^2 f}{dx^2}$, $f^{(n)}(x) = \dfrac{d^n f}{dx^n}$

Example $s(t)$ = position $\Rightarrow$ $s'(t)$ = speed $\Rightarrow$ $s''(t)$ acceleration.

Example $f(x) = 3e^{(x^2)}$ $\Rightarrow$ $f'(x) = 3e^{(x^2)} \cdot 2x$ Chain Rule

$f'(x) = u(x)\,v(x)$, $u(x) = 3e^{(x^2)}$, $v(x) = 2x$

$\Rightarrow u'(x) = 3e^{(x^2)} \cdot 2x$, $v'(x) = 2$

$\Rightarrow f''(x) = u'(x)v(x) + u(x)v'(x) = 3e^{(x^2)} \cdot (2x) \cdot (2x) + 3e^{(x^2)} \cdot 2$.

Conclusion (Concavity Test)

$f''(x) > 0$ on (a,b) $\Rightarrow$ f' increasing on (a,b) $\Rightarrow$ f concave up on (a,b)

$f''(x) < 0$ on (a,b) $\Rightarrow$ f' decreasing on (a,b) $\Rightarrow$ f concave down on (a,b)

Remark We can determine the concavity of function by doing sign analysis on f''.

Example $f(x) = x^3 - x \Rightarrow f'(x) = 3x^2 - 1 \Rightarrow f''(x) = 6x$

A/ $f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$

B/ f'' continuous everywhere

$$\begin{array}{ccccc} & - & 0 & + & f''(x) \end{array}$$

$f''(-1) = 6\cdot(-1) = -6 < 0 \quad f''(1) = 6\cdot 1 = 6 > 0$

$\Rightarrow$ f concave down on $(-\infty, 0)$.

f concave up on $(0, \infty)$

f differentiable at $0 \Rightarrow (0,0)$ is an inflection point on $y = x^3 - x$.

Example Determine where $y = x^4 - 8x^3 + 18x^2$ is concave up/down. Also determine the inflection points.

$f(x) = x^4 - 8x^3 + 18x^2 \Rightarrow f'(x) = 4x^3 - 24x^2 + 36x$

$\Rightarrow f''(x) = 12x^2 - 48x + 36 = 12(x^2 - 4x + 3) = 12(x-1)(x-3)$

A/ $f''(x) = 0 \Rightarrow 12(x-1)(x-3) = 0 \Rightarrow x = 1, 3$

B/ f'' continuous everywhere.

$$\begin{array}{cccccc} + & 1 & - & 3 & + & f''(x) \end{array}$$

$f(0) = 12(-1)(-3) > 0 \quad f''(2) = 12\cdot 1\cdot(-1) < 0 \quad f''(4) = 12\cdot 3\cdot 1 > 0$

$\Rightarrow$ f concave up on $(-\infty, 1)$ and $(3, \infty)$

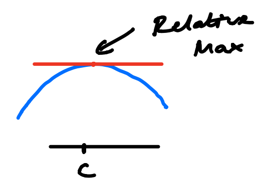
f concave down on $(1,3)$

$(1, f(1)), (3, f(3))$

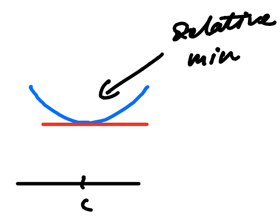
$f'(1)$ and $f'(3)$ exist $\Rightarrow (1, 11), (3, 27)$ inflection points

Q_1: Can we use concavity to find relative max / min points?

$f'(c) = 0$
$f''(c) < 0$ $\Rightarrow$ Tangent line horizontal at $x = c$ **and** Concave down at $x=c$ $\Rightarrow$



$f'(c) = 0$
$f''(c) > 0$ $\Rightarrow$ Tangent line horizontal at $x = c$ **and** Concave up at $x = c$ $\Rightarrow$



Second Derivative Test for Relative Extrema

Let f'' exist on some open interval containing c (except maybe at c). Then

1/ $f''(c) > 0$ **and** $f'(c) = 0$ $\Rightarrow$ $f(c)$ relative min

2/ $f''(c) < 0$ **and** $f'(c) = 0$ $\Rightarrow$ $f(c)$ relative max

3/ If $f''(c) = 0$ or DNE we cannot conclude anything.

Example $f(x) = x^3 - x \Rightarrow f'(x) = 3x^2 - 1 \Rightarrow f''(x) = 6x$

$f'(x) = 0 \Rightarrow 3x^2 - 1 = x = \pm\sqrt{\frac{1}{3}}$

$f'(\sqrt{\frac{1}{3}}) = 6 \cdot \sqrt{\frac{1}{3}} > 0$ $\left(\text{and } f'(\sqrt{\frac{1}{3}}) = 0\right)$

$f'(-\sqrt{\frac{1}{3}}) = 6 \cdot (-\sqrt{\frac{1}{3}}) < 0$ $\left(\text{and } f'(-\sqrt{\frac{1}{3}}) = 0\right)$

$\Rightarrow f(\sqrt{\frac{1}{3}})$ relative min and $f(-\sqrt{\frac{1}{3}})$ relative max.

Remark The 1st derivative test is more powerful as we do not need $f'(c) = 0$ and $f''(c) \neq 0$. The 2nd derivative test will not generally pick up all relative extrema.

Example $f(x) = x^4$

2^{nd} Derivative Approach :

$f(x) = x^4 \Rightarrow f'(x) = 4x^3 \Rightarrow f''(x) = 12x^2$

$f'(x) = 0 \Rightarrow 4x^3 = 0 \Rightarrow x = 0$

$f''(0) = 12 \cdot 0^2 = 0$

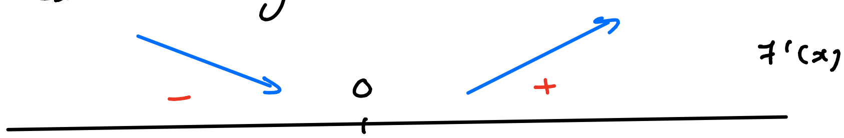
$\Rightarrow$ 2^{nd} Derivative Test is inconclusive

1^{st} Derivative Approach :

1/ $f'(x) = 4x^3$

2/ A/ $f'(x) = 0 \Rightarrow x = 0$

B/ f' continuous everywhere



$f(-1) = 4 \cdot (-1)^3 < 0$ $\quad$ $f'(1) = 4 \cdot 1^3 > 0$

3/ f continuous at 0 $\Rightarrow$ $f(0)$ relative min