Antiderivatives

F, f - functions

$F'(x) = f(x) \implies f$ is derivative of F

Alternate perspective : F is an <u>antiderivative</u> of f.

<u>Examples</u> :

1/ Speed = Derivative of position (with respect to time)

Position = Antiderivative of speed

2/ $\frac{d}{dx}(x^2) = 2x \implies x^2$ an antiderivative of $2x$

3/ $\frac{d}{dx}(\ln|x|) = \frac{1}{x} \implies \ln|x|$ an antiderivative of $\frac{1}{x}$

3/ $\frac{d}{dx}(e^x) = \frac{d}{dx}(e^x + 1) = e^x$

$\implies$ Both e^x <u>and</u> $e^x + 1$ are antiderivatives of e^x

Important Observation : Derivatives are unique

Antiderivatives are <u>not</u>.

<u>Fact</u> : F, G both antiderivatives of f on interval $\implies G(x) = F(x) + C$ for some constant C, for all x in interval.

Example x^2 an antiderivative of $2x$

$\Rightarrow$ Every antiderivative of $2x$ is of form $x^2 + C$ for some constant C.

Important Definition/Notation (Indefinite Integral)

called the indefinite integral of f with respect to x

$$\int f(x)\, dx = \text{General Antiderivative of } f(x)$$

"Integral symbol"

"Integrand"

Indicates that the independent variable is x. Just notation.
dx sometimes called the differential.

Fact + Definition $\Rightarrow$ $\int f(x)\,dx = F(x) + C$

where $F(x)$ is any fixed antiderivative of $f(x)$ (ie $F'(x) = f(x)$)

Examples $\int 2x\, dx = x^2 + C$, $\int e^x\, dx = e^x + C$

Q: What is an antiderivative of x^n, a power function?

Case 1 ($n \neq -1$)

$$\frac{d}{dx}\left(x^{(n+1)}\right) = (n+1)\, x^n$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{(n+1)} x^{(n+1)}\right) = x^n$$

If $n = -1$ this would be 0.

$\Rightarrow \frac{1}{n+1} x^{n+1}$ an antiderivative of x^n

Case 2 $(n = -1)$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} = x^{-1} \Rightarrow \ln|x| \text{ an antiderivative of } x^{-1}$$

Conclusion :

$$\boxed{\int x^n\,dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{if } n \neq -1 \\ \ln|x| + C & \text{if } n = -1 \end{cases}}$$

Q, : What is an antiderivative of a^x, an exponential function ?

$$\frac{d}{dx}(a^x) = \ln(a)\,a^x \Rightarrow \frac{d}{dx}\left(\frac{1}{\ln(a)}a^x\right) = \frac{1}{\ln(a)}\ln(a)\,a^x = a^x$$

$\Rightarrow \dfrac{a^x}{\ln(a)}$ is an antiderivative of a^x

$$\boxed{\int a^x\,dx = \frac{1}{\ln(a)}a^x + C \Rightarrow \int e^x\,dx = e^x + C}$$

Remark

1/ You'll have to wait until 16B to see how to find antiderivatives of $\log_a(x)$.

2/ Our laws of differentiation give rise to laws for finding indefinite integrals (antiderivatives)

Sum / Difference Law

$$\int f(x)\,dx = F(x) + C$$

$$\int g(x)\,dx = G(x) + C$$

$$\Rightarrow \int (f(x) \pm g(x))\,dx = F(x) + G(x) + C$$

(C an arbitrary constant)

Constant Multiple Law

$$\int f(x)\,dx = F(x) + C \Rightarrow \int k f(x)\,dx = k F(x) + C$$

(C an arbitrary constant)

Examples

1/ $\int (x^2 + 3x + 1)\,dx = ?$

$$\int x^2 dx = \frac{1}{3} x^3 + C$$

$$\int 3x\,dx = 3 \cdot \frac{1}{2} x^2 + C$$

$$\int \underset{= x^0}{1}\,dx = \frac{1}{0+1} x^{0+1} + C = x + C$$

$$\Rightarrow \int (x^2 + 3x + 1)\,dx = \frac{1}{3} x^3 + \frac{3}{2} x^2 + x + C$$

2/ $\int \frac{x^2 - 1}{\sqrt{x}}\,dx = ?$

$$\frac{x^2 - 1}{\sqrt{x}} = x^{3/2} - x^{-\frac{1}{2}}$$

$$\int x^{3/2}\,dx = \frac{1}{3/2 + 1} \cdot x^{3/2 + 1} + C = \frac{2}{5} x^{5/2} + C$$

$$\int x^{-\frac{1}{2}}\,dx = \frac{1}{-\frac{1}{2} + 1} x^{-\frac{1}{2} + 1} + C = 2x^{1/2} + C$$

$$\Rightarrow \int \frac{x^2-1}{\sqrt{x}}\,dx = \frac{2}{5}x^{5/2} - 2x^{1/2} + C$$

3/ Find an antiderivative $F(t)$ of $2e^t - t^{-1}$ such that $F(1) = 0$.

$$\int e^t\,dt = e^t + C \quad \Rightarrow \quad \int 2e^t\,dt = 2e^t + C$$

$$\int t^{-1}\,dt = \ln|t| + C$$

$$\Rightarrow \int 2e^t - t^{-1}\,dt = 2e^t - \ln|t| + C$$

Need to find C such that $2e^1 + \ln|1| + C = 0$

$$\Rightarrow C = -2e$$

$\Rightarrow F(t) = 2e^t - \ln|t| - 2e$ is desired antiderivative.