Permutation Groups and Group Actions

Let S be a set.

Definition

The **permutation group** of S is set $\Sigma(S) := \{f : S \to S \mid f \text{ bijection}\}$,

equipped with function composition, i.e. $(f \circ g)(x) = f(g(x))$ $\forall x \in S$.

Important Exercise : Prove $(\Sigma(S), \circ)$ is a group. $\left(\begin{array}{l}\text{e.g.} \\ e = Id_S\end{array}\right)$

Important Example : $Sym_n := \sum(\{1, 2, \dots, n\})$ $\left(\begin{array}{l}|S| = n \\ \Rightarrow \Sigma(S) \cong Sym_n\end{array}\right)$

n^{th} symmetric group

Observation : There is a map $\mu : \Sigma(S) \times S \longrightarrow S$
$$(f, x) \longmapsto f(x)$$

Properties :
1) $Id_S(x) = x \quad \forall x \in S$
2) $\forall f, g \in \Sigma(S)$, $(f \circ g)(x) = f(g(x))$ $\forall x \in S$

moves around bijectively

Intuition : An element $f \in \Sigma(S)$ permutes / acts on the set S.

Definition Let $(G, *)$ be a group. An action of G on S

is a map $\mu : G \times S \longrightarrow S$
$$(g, x) \longmapsto g(x)$$

g not a function in general. This is just notation for $\mu(g, s)$

such that
1) $e(x) = x \quad \forall x \in S$
2) $\forall g, h \in G$, $(g * h)(x) = g(h(x))$ $\forall x \in S$

Intuition : Elements of G permute S in a way compatible with the group structure $*$.

G can act on itself in several ways :

- (Left Regular Representation)

$\mu : G \times G \longrightarrow G$
$(g, x) \longmapsto g * x$

1) $e(x) = e * x = x \quad \forall x \in G$
2) $(g * h)(x) = (g * h) * x = g * (h * x) = g(h(x))$

- (Conjugation)

$$\mu : G \times G \longrightarrow G$$
$$(g, x) \longmapsto g * x * g^{-1}$$

1/ $e(x) = e * x * e^{-1} = x \quad \forall x \in G$

2/ $(g*h)(x) = (g*h) * x * (g*h)^{-1}$
$= g * h * x * h^{-1} * g$
$= g * (h * x * h^{-1}) * g^{-1}$
$= g(h(x))$

<u>Theorem</u> Giving an action of G on S is the same as giving a homomorphism $\phi : G \longrightarrow \Sigma(S)$.

<u>Proof</u> (Outline)

$$\left\{ \begin{array}{l} \mu : G \times S \longrightarrow S \;\Big|\; \\ 1/\; e(x) = x \quad \forall x \\ 2/\; (f*g)(x) = f(g(x)) \\ \quad \forall f, g \in G, x \in S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \phi : G \longrightarrow \Sigma(S) \;\Big| \\ \phi(f*g) = \phi(f) \circ \phi(g) \\ \forall f, g \in G \end{array} \right\}$$

$$\mu \longmapsto \phi_\mu : G \longrightarrow \Sigma(S)$$
$$g \longrightarrow \phi_\mu(g) : S \longrightarrow S$$
$$x \longrightarrow g(x)$$

Inverses to each other

2/ ⇒ homomorphism

1/, 2/ ⇒ bijection

$\phi_\mu(g^{-1})$ is inverse map

$$\mu_\phi : G \times S \longrightarrow S \qquad \longleftarrow\!\!\!| \qquad \phi$$
$$(g, x) \longrightarrow \phi(g)(x)$$

ϕ homomorphism ⇒ 1/ and 2/

$\square$

<u>Proposition</u> Let $(G, *)$, $(H, \circ)$ be groups and $\phi : G \longrightarrow H$ a homomorphism. Then $Im(\phi) \subset H$ is a subgroup.

<u>Proof</u>

- $\phi(e_G) = e_H \Rightarrow e_H \in Im(\phi)$
- Let $x, y \in Im(\phi) \Rightarrow \exists\, x', y' \in G$ such that $\phi(x') = x$, $\phi(y') = y$
  $\Rightarrow x \circ y = \phi(x') \circ \phi(y') = \phi(x' * y') \Rightarrow x \circ y \in Im(\phi)$
- Let $x \in Im(\phi) \Rightarrow \exists\, x' \in G$ such that $\phi(x') = x$
  $\Rightarrow x^{-1} = (\phi(x'))^{-1} = \phi((x')^{-1}) \Rightarrow x^{-1} \in Im(\phi)$ $\square$

Definition

We say an action $\phi : G \to \Sigma(S)$ is faithful if ϕ injective.

Observe that $\phi : G \to \Sigma(S)$ injective $\Rightarrow G \cong \mathrm{Im}(\phi)$

$\Rightarrow G$ isomorphic to subgroup of $\Sigma(S)$.

Proposition The left regular representation is faithful.

Proof

$$\mu : G \times G \to G, \quad (g, x) \mapsto g * x \quad \longrightarrow \quad \phi_\mu : G \to \Sigma(G), \quad g \to \phi_\mu(g) : G \to G, \ x \to g * x$$

Let $h, g \in G$ such that $\phi_\mu(h) = \phi_\mu(g)$

$\Rightarrow g * x = h * x \quad \forall x \in G$

$\Rightarrow g * e = h * e \Rightarrow g = h$

$\square$

Group theory is really about symmetry

Cayley's Theorem Any group G is isomorphic to a subgroup of a permutation group. If $|G| = n \in \mathbb{N} \Rightarrow G$ isomorphic to a subgroup of Sym_n.

So these groups contain all finite groups

Proof Let μ be the left regular representation of G on G.

$\Rightarrow \phi_\mu : G \to \Sigma(G)$ injective

$\Rightarrow G \cong \mathrm{Im}(\phi_\mu) \subset \Sigma(G)$

$|G| = n \Rightarrow G \cong \mathrm{Im}(\phi_\mu) \subset \Sigma(G) \cong Sym_n$

$\square$