Homework 8 Solutions

1/ Absolute max at x_3 ; No absolute min

3/ No absolute extrema

7/ Absolute max at x_1 ; Absolute min at x_2

8/ Absolute max at x_2 ; Absolute min at x_1

11/ $f(x) = x^3 - 6x^2 + 9x - 8$ on $[0,5]$

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

A/ $f'(x) = 0 \Rightarrow x = 1, 3$

B/ f' continuous everywhere

$\Rightarrow$ $0, 1, 3, 5$ = critical numbers on $[0,5]$

$f(0) = -8$

$f(1) = -4$

$f(3) = -8$

$f(5) = 12$

$\Rightarrow$ Absolute max of 12 at $x = 5$

Absolute min of -8 at $x = 0, 3$

15/ $f(x) = x^4 - 18x^2 + 1$ on $[-4,4]$

$f'(x) = 4x^3 - 36x = 4x(x^2-9) = 4x(x+3)(x-3)$

A/ $f'(x) = 0 \Rightarrow x = 0, \pm 3$

B/ f' continuous everywhere

$\Rightarrow$ $0, \pm 3, \pm 4$ = critical numbers on $[-4,4]$

$f(0) = 1$

$f(3) = f(-3) = -80$

$f(4) = f(-4) = -32$

$\Rightarrow$ Absolute max of 1 at $x = 0$

Absolute min of -80 at $x = \pm 3$.

19/ $f(x) = \dfrac{x-1}{x^2+1}$ on $[1,5]$

$$f'(x) = \frac{\frac{d(x-1)}{dx}(x^2+1) - (x-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - (x-1)(2x)}{(x^2+1)^2} = \frac{-x^2+2x+1}{(x^2+1)^2}$$

A/ $f'(x) = 0 \Rightarrow -x^2+2x+1 = 0$

$$\Rightarrow x = \frac{-2 \pm \sqrt{2^2-4(-1)\cdot 1}}{-2} = 1 \pm \sqrt{2}$$

B/ $x^2+1 > 0 \Rightarrow f'$ continuous everywhere

$\Rightarrow 1,\ 1+\sqrt{2},\ 5$ are critical numbers on $[1,5]$

$f(1) = 0$

$f(1+\sqrt{2}) \simeq 0.21$ $\Rightarrow$ Absolut max $A \simeq 0.21$ at $x = 1+\sqrt{2}$

Absolute min 0 at $x = 1$

$f(5) = \dfrac{4}{26}$

23 $f(x) = 5x^{2/3} + 2x^{5/3}$ on $[-2,1]$

$f'(x) = \dfrac{10}{3}\dfrac{1}{x^{1/3}} + \dfrac{10}{3}x^{2/3}$

A/ $f'(x) = 0 \Rightarrow \dfrac{10}{3}\dfrac{1}{x^{1/3}} = \dfrac{-10}{3}x^{2/3} \Rightarrow -1 = x$

B/ f' undefined at 0

$\Rightarrow -2, -1, 0, 1$ are critical numbers on $[-2,1]$

$f(-2) = 5\cdot(-2)^{2/3} + 2\cdot(-2)^{5/3}$ ↑ check on calculator

$f(-1) = 5-2 = 3$

$f(0) = 0$

$f(1) = 7$

$\Rightarrow$ Absolute max of 7 at $x=1$

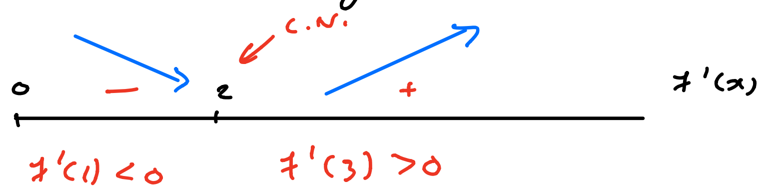
Absolute min of 0 at $x=0$

31/ $f(x) = 2x + \frac{8}{x^2} + 1$ on $(0,\infty)$

$f'(x) = 2 - \frac{16}{x^3}$

A/ $f'(x) = 0 \Rightarrow 2 - \frac{16}{x^3} = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$

B/ f' continuous everywhere on $(0,\infty)$



$\Rightarrow f(2) = 7$ absolute max.

There is a vertical asymptote at 0 so no absolute max.

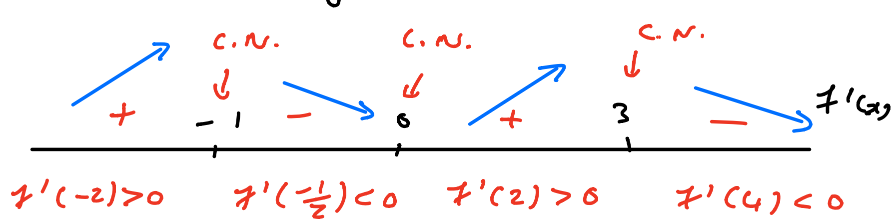
33/ $f(x) = -3x^4 + 8x^3 + 18x^2 + 2$ on $(-\infty,\infty)$

$f'(x) = -12x^3 + 24x^2 + 36x = -12x(x^2 - 2x - 3)$

$= -12x(x-3)(x+1)$

A/ $f'(x) = 0 \Rightarrow x = 0, 3, -1$

B/ f' continuous everywhere



$\Rightarrow$ $f(-1)$ or $f(3)$ absolute max

$f(-1) < f(3) = 137$. $\Rightarrow$ 137 absolute max at $x = 3$

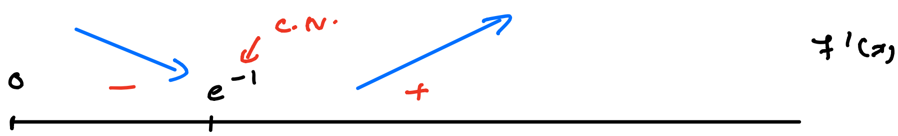
$\lim_{x \to \infty} f(x) = -\infty$ $\Rightarrow$ No absolute min

38/ $f(x) = x \ln(x)$ on $(0, \infty)$

$f'(x) = 1 \cdot \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$

A/ $f'(x) = 0 \Rightarrow \ln(x) + 1 = 0 \Rightarrow x = e^{-1}$

B/ f' continuous on $(0, \infty)$



$f'(e^{-2}) = -1 < 0$ $\quad f'(e) = 2 > 0$

$\Rightarrow$ $e^{-1}(-1) =$ absolute min at $x = e^{-1}$

$\lim_{x \to \infty} x \ln(x) = \infty$ $\Rightarrow$ No absolute max.

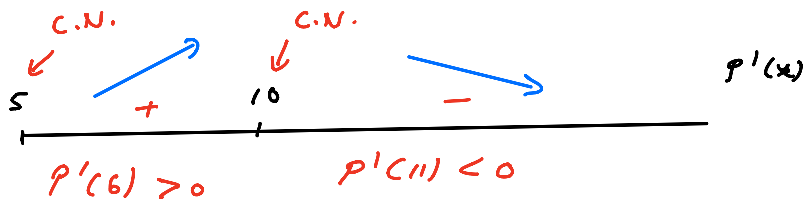
43/ $P(x) = -x^3 + 9x^2 + 120x - 400$ on $[5, \infty)$

$P'(x) = -3x^2 + 18x + 120 = -3(x^2 - 6x - 40)$

$= -3(x - 10)(x + 4)$

A/ $P'(x) = 0 \Rightarrow x = 10$ or -4

B/ P' continuous everywhere



$\Rightarrow \quad P(10) = \text{absolute max}$

$\quad\quad\;\; \overset{\shortparallel}{700}$

$\Rightarrow$ Max profit is \$700,000 when 1000,000 tires are sold

45/ $\quad C(x) = x^3 + 37x + 250$

$\Rightarrow \quad \overline{C}(x) = x^2 + 37 + \dfrac{250}{x}$

$\overline{C}'(x) = 2x - \dfrac{250}{x^2}$

A/ $\quad \overline{C}'(x) = 0 \Rightarrow x^3 = 125 \Rightarrow x = 5$

B/ $\quad \overline{C}'$ undefined $\Rightarrow x = 0$

a) $$\left.\begin{aligned} \overline{C}(1) &= 288 \\ \overline{C}(5) &= 25 + 37 + 50 = 112 \\ \overline{C}(10) &= 100 + 37 + 25 = 162 \end{aligned}\right\} \text{ 112 min value on } [1, 10]$$

b) $\quad \overline{C}(20) = 400 + 37 + 12.5 > 162$

$\Rightarrow$ 162 is min value on $[10, 20]$

53/ $$f'(x) = \frac{\frac{d}{dx}(x^2+36) \cdot 2x - (x^2+36) \cdot \frac{d}{dx}(2x)}{(2x)^2}$$

$$= \frac{(2x)\cdot(2x) - 2(x^2+36)}{4x^2} = \frac{2x^2-72}{4x^2}$$

A/ $f'(x) = 0 \Rightarrow 2x^2 - 72 = 0 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

B/ f' undefined $\Rightarrow x = 0$

$$\left.\begin{aligned} f(1) &= \frac{37}{2} \\ f(6) &= 6 \\ f(12) &= \frac{144+36}{24} \end{aligned}\right\}$$ The minimum percentage on $[1, 12]$ is 6% at 6 months.

§6.2 / 3, 7, 9, 13, 19, 23, 32, 33, 51

3/ Objective : maximize x^2y

Constraint : $x + y = 90$, $x, y \geq 0$

$\Rightarrow y = 90 - x \Rightarrow x^2y = x^2(90 - x) = 90x^2 - x^3 = f(x)$

Domain $x \geq 0$, $x \leq 90$ $= [0, 90]$

$f'(x) = 180x - 3x^2$

A/ $f'(x) = 0 \Rightarrow x(180 - 3x) = 0 \Rightarrow x = 0, 60$

B/ f' continuous on $[0, 90]$

$f(0) = 0$
$f(60) = 108000$ $\Rightarrow f(60)$ absolute max on $[0, 90]$
$f(90) = 0$

$x = 60 \Rightarrow y = 30$

$\Rightarrow \quad x^2y$ is maximized when $x = 60,\ y = 30$. The max value is 108000.

7/ a) $R(x) = p(x) \cdot (1000\,x)$ ← number of units sold

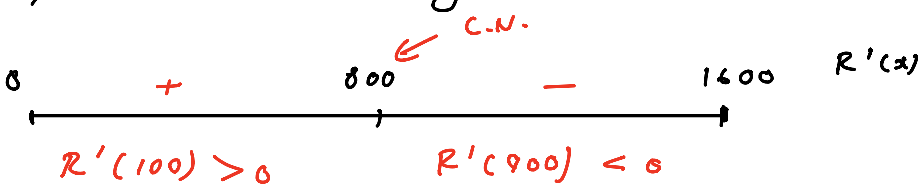
$$= \left(160 - \frac{x}{10}\right)(1000\,x) = 160000\,x - 100\,x^2$$

b)/c) $R'(x) = 160000 - 200\,x$

A/ $R'(x) = 0 \Rightarrow x = 800$

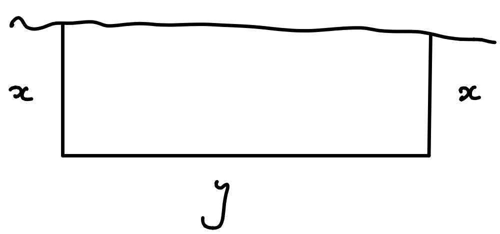
B/ R' continuous everywhere

$$\left(\begin{array}{l} p(x) \geq 0 \\ \Rightarrow 160 - \dfrac{x}{10} \geq 0 \\ \Rightarrow x \leq 1600 \end{array}\right.$$



$\Rightarrow \quad R(800) = 640\,000$ is absolute max revenue.

9/ a)



$$2x + y = 1400 \Rightarrow y = 1400 - 2x$$

b) Area $= xy = 1400x - 2x^2 = f(x)$

c)/(d) $x \geq 0,\ y \geq 0,\quad 2x + y = 1400 \Rightarrow x \leq 700$

<u>Domain</u> $= [0, 700]$

$f'(x) = 1400 - 4x$

A/ $f'(x) = 0 \Rightarrow x = 350$

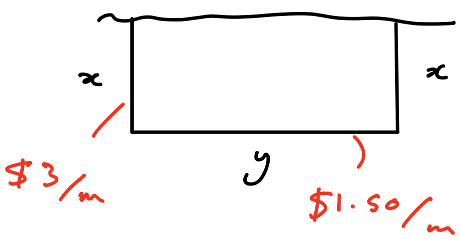
B/ f' continuous on $[0,700]$

$f(0) = 0$

$f(350) = 245{,}000$ $\Rightarrow$ Max area is $245{,}000$ when $x = 350$.

$f(700) = 0$

13/ Objective : Minimize <u>Cost</u>



$\Rightarrow$ Cost $= 3x + 1.5y + 3x$

$= 6x + \frac{3}{2}y$ ← objective

<u>Constraint</u> : Area $= 25600\ m^2$

$\Rightarrow xy = 25600 \Rightarrow y = \dfrac{25600}{x}$

$\Rightarrow 6x + \frac{3}{2}y = 6x + \frac{3}{2}\cdot\dfrac{25600}{x} = f(x)$

<u>Domain</u> : $x \neq 0,\ x \geq 0 = (0, \infty)$

$f'(x) = 6 - \frac{3}{2}\cdot\dfrac{25600}{x^2}$

A/ $f'(x) = 0 \Rightarrow x^2 = \dfrac{\frac{3}{2}\times 25600}{6} = \dfrac{25600}{4}$

$\Rightarrow x = \pm\dfrac{160}{2} = \pm 8$

B/ f' continuous on $(0, \infty)$

← C.N.

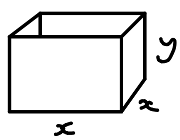
0 — 8 + $f'(x)$

$f'(1) < 0$ $f'(9) > 0$

$\Rightarrow f(8) = 960$ is absolute min on $(0, \infty)$

$\Rightarrow$ Minimal cost is $\$960$.

19/ Objective : Minimize Surface area



Surface area $= x^2 + 4xy$ ← objective

Constraint : Volume $= 32$ ← constraint

$\Rightarrow x^2 y = 32 \Rightarrow y = \frac{32}{x^2}$

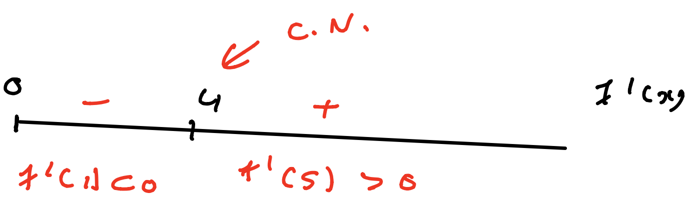
$\Rightarrow x^2 + 4xy = x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x} = f(x)$

Domain : $x \neq 0$, $x > 0$ $= (0, \infty)$

$f'(x) = 2x - \frac{128}{x^2}$

A/ $f'(x) = 0 \Rightarrow 2x = \frac{128}{x^2} = x^3 = 64 \Rightarrow x = 4$

B/ f' continuous on $(0, \infty)$

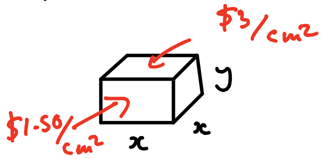


$\Rightarrow f(4)$ absolute min on $(0, \infty)$

$x = 4 \Rightarrow y = 2$

$\Rightarrow$ Min surface area is when $x = 4$ and $y = 2$.

23/ Objective : minimize Cost



$\Rightarrow$ Cost $= 3x^2 + 3x^2 + 1.5xy + 1.5xy + 1.5xy$
$+ 1.5xy$ ← objective
$= 6x^2 + 6xy$

Constraint : Volume = 16000

$$\Rightarrow x^2 y = 16000 \Rightarrow y = \frac{16000}{x^2}$$

$$\Rightarrow 6x^2 + 6xy = 6x^2 + 6x \cdot \frac{16000}{x^2}$$

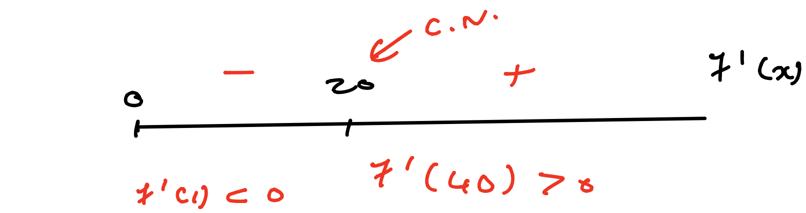
$$= 6x^2 + \frac{6 \times 16000}{x} = f(x)$$

Domain : $x \neq 0,\ x \geq 0 = (0, \infty)$

$$f'(x) = 12x - \frac{6 \times 16000}{x^2}$$

A/ $f'(x) = 0 \Rightarrow x^3 = 8000 \Rightarrow x = 20$

B/ f' continuous on $(0, \infty)$

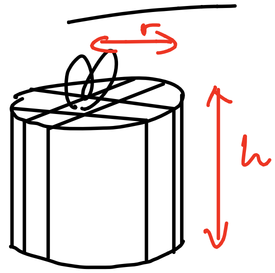


$\Rightarrow f(20) = 7200$ absolute min on $(0, \infty)$

$x = 20 \Rightarrow y = 40$

$\Rightarrow$ Min cost is \$7200 when $x = 20$cm, $y = 40$cm.

32/ Objective : Maximize Volume



objective

$$\text{Volume} = \pi r^2 h$$

Constraint : Ribbon length = 130

$$\Rightarrow 10 + 4r + 4h + 4r = 130$$

$\Rightarrow \quad 8r + 4h = 120$ ← constraint

$\Rightarrow \quad 2r + h = 30 \quad \Rightarrow \quad h = 30 - 2r$

$\Rightarrow \quad \pi r^2 h = \pi r^2 (30 - 2r) = 30\pi r^2 - 2\pi r^3 = f(r)$

Domain : $r \geq 0$, $h \geq 0$, $2r + h = 30 \Rightarrow r \leq 15$

$[0, 15]$

$f'(r) = 60\pi r - 6\pi r^2$

A/ $f'(r) = 0 \Rightarrow 6\pi r(10 - r) = 0 \Rightarrow r = 0, 10$

B/ f' continuous on $[0, 15]$

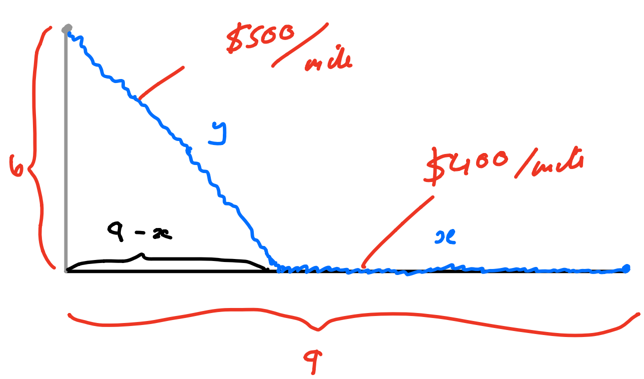
$f(0) = 0$

$f(15) = 0 \quad \Rightarrow \quad f(10)$ absolute max on $[0, 15]$

$f(10) > 0$

$r = 10 \Rightarrow h = 10 \Rightarrow$ Max volume occurs when $r = 10$ cm , $h = 10$ cm

33/ Objective : Minimize Cost



Cost $= 400x + 500y$

Constraint : $6^2 + (9 - x)^2 = y^2$

$\Rightarrow y = \sqrt{6^2 + (9 - x)^2}$

$$\Rightarrow \quad 400x + 500y = 400x + 500\sqrt{6^2+(9-x)^2} = f(x)$$

Domain $\quad x \geq 0, \quad x \leq 9 \quad = [0,9]$

$$f'(x) = 400 + 500 \cdot \frac{1}{2}\left(6^2+(9-x)^2\right)^{-\frac{1}{2}} \cdot 2(9-x)\cdot(-1)$$

$$= 400 - \frac{500(9-x)}{\sqrt{6^2+(9-x)^2}}$$

A/ $\quad f'(x) = 0 \quad \Rightarrow 500(9-x) = 400\sqrt{6^2+(9-x)^2}$

$$\Rightarrow \quad 500^2(9-x)^2 = 400^2\left(6^2+(9-x)^2\right)$$

$$\Rightarrow \quad (500^2-400^2)(9-x)^2 = 400^2 \cdot 6^2$$

$$\Rightarrow \quad (9-x)^2 = \frac{400^2 \cdot 6^2}{500^2-400^2} = 64$$

$$\Rightarrow \quad 9-x = 8 \quad \Rightarrow \quad x = 1$$

B/ $\quad f'$ continuous on $[0,9]$

$$f(0) = 500\sqrt{6^2+9^2}$$

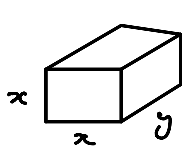
$$f(9) = 400 \cdot 9 + 500\sqrt{6^2}$$

minimum value

$$f(1) = 400 + 500\sqrt{6^2+8^2}$$

$\Rightarrow$ Cost minimized when $x = 1$.

51/ $\quad$ Objective : Maximize Volume



Volume $= x^2 y$

Constraint : $\quad$ Girth + length $= 108$

$\Rightarrow \quad 4x + y = 108 \quad \Rightarrow \quad y = 108 - 4x$

$\Rightarrow \quad x^2 y = 108x^2 - 4x^3 = f(x)$

<u>Domain</u> $\quad x, y \geq 0, \quad 4x + y = 108 \Rightarrow x \leq \frac{108}{4} = 27$

$[0, 27]$

$f'(x) = 216x - 12x^2$

A/ $f'(x) = 0 \Rightarrow x = 0$ or 18

B/ f' continuous on $[0, 27]$

$f(0) = 0$
$f(27) = 0$ $\quad \Rightarrow \quad$ Volume is maximized when $x = 18$, $y = 36$
$f(18) > 0$