Homework 7 Solutions

§5.3

4/ $f(x) = -x^4 + 7x^3 - \frac{x^2}{2} \Rightarrow f'(x) = -4x^3 + 21x^2 - x$

$\Rightarrow f''(x) = -12x^2 + 42x - 1$

$\Rightarrow f''(0) = -1 \quad , \quad f''(2) = 35$

9/ $f(x) = (x^2+4)^{\frac{1}{2}} \Rightarrow f'(x) = \underbrace{\frac{1}{2}(x^2+4)^{-\frac{1}{2}}}_{u} \cdot \underbrace{2x}_{v}$

$u'(x) = \frac{-1}{4}(x^2+4)^{-\frac{3}{2}} \cdot 2x \quad , \quad v'(x) = 2$

$\Rightarrow f''(x) = u'(x)\,v(x) + u(x)\,v'(x)$

$= \frac{-1}{4}(x^2+4)^{-\frac{3}{2}} \cdot 2x \cdot 2x + \frac{1}{2}(x^2+4)^{-\frac{1}{2}} \cdot 2$

$\Rightarrow f''(0) = \frac{1}{2} \quad , \quad f''(2) = \frac{1}{4\sqrt{2}}$

15/ $f(x) = \frac{u(x)}{v(x)}$, $u(x) = \ln(x)$, $v(x) = 4x \Rightarrow u'(x) = \frac{1}{x}$, $v'(x) = 4$

$\Rightarrow f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} = \frac{\frac{1}{x} \cdot 4x - \ln(x) \cdot 4}{16x^2}$

$= \frac{4(1 - \ln(x))}{16x^2}$

$\Rightarrow f''(x) = \frac{\frac{d}{dx}(4(1-\ln(x)))\,16x^2 - 4(1-\ln(x))\frac{d}{dx}(16x^2)}{(16x^2)^2}$

$= \frac{\left(\frac{-4}{x} \cdot 16x^2\right) - \left(4(1-\ln(x)) \cdot 32x\right)}{(16x^2)^2}$

$\Rightarrow f''(0)$ DNE , $f''(2) \approx -0.05$

23/ $f(x) = \dfrac{u(x)}{v(x)}$, $u(x) = 3x$, $v(x) = x-2 \Rightarrow u'(x) = 3, v'(x) = 1$

$$\Rightarrow f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} = \frac{3(x-2) - 3x \cdot 1}{(x-2)^2}$$

$$= -6(x-2)^{-2}$$

$$\Rightarrow f''(x) = 12(x-2)^{-3} \cdot 1$$

$$\Rightarrow f'''(x) = -36(x-2)^{-4} \cdot 1$$

$$\Rightarrow f^{(4)}(x) = 144(x-2)^{-5} \cdot 1$$

29/ Concave Up : $(2, \infty)$

Concave Down : $(-\infty, 2)$

Inflection : $(2, 3)$

31/ Concave Up : $(-\infty, -1)$ and $(8, \infty)$

Concave Down : $(-1, 8)$

Inflection : $(-1, 7)$ and $(8, 6)$

34/ Concave Up : $(-\infty, 0)$

Concave Down : $(0, \infty)$

Inflection : None

37/ $f(x) = -2x^3 + 9x^2 + 168x - 3$

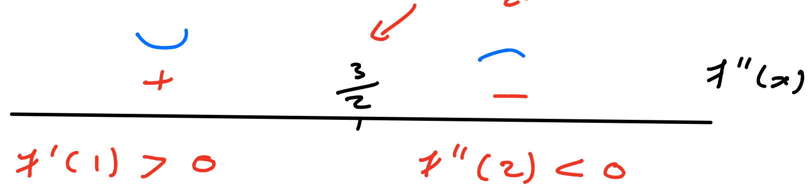
$\Rightarrow f'(x) = -6x^2 + 18x + 168$

$\Rightarrow f''(x) = -12x + 18$

A/ $f''(x) = 0 \Rightarrow -12x + 18 = 0 \Rightarrow x = \frac{3}{2}$

B/ f'' continuous everywhere

$f'(\frac{3}{2})$ exists



$\Rightarrow$ Concave Up : $(-\infty, \frac{3}{2})$

Concave Down : $(\frac{3}{2}, \infty)$

Inflection : $(\frac{3}{2}, f(\frac{3}{2}))$

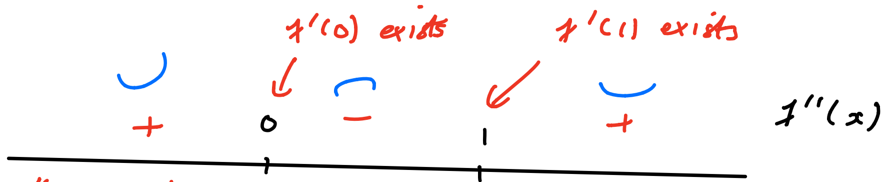
45/ $f(x) = x^{8/3} - 4x^{5/3}$

$\Rightarrow f'(x) = \frac{8}{3}x^{5/3} - \frac{20}{3}x^{2/3}$

$\Rightarrow f''(x) = \frac{40}{9}x^{2/3} - \frac{40}{9}x^{-\frac{1}{3}} = \frac{40}{9}x^{2/3} - \frac{40}{9}\frac{1}{x^{1/3}}$

A/ $f''(x) = 0 \Rightarrow \frac{40}{9}x^{2/3} = \frac{40}{9} \cdot \frac{1}{x^{1/3}} \Rightarrow x = 1$

B/ f'' undefined $\Rightarrow x = 0$



$f''(-1) = \frac{40}{9} + \frac{40}{9} > 0$ $\qquad f''(8) > 0$

$\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{1}{2}$ $\qquad f''\left(\frac{1}{8}\right) = \frac{40}{9}\left(\frac{1}{2}\right)^2 - \frac{40}{9}\frac{1}{\left(\frac{1}{2}\right)} < 0$

$(8)^{1/3} = 2$

$\Rightarrow$ Concave Up : $(-\infty, 0)$ and $(1, \infty)$

Concave Down : $(0, 1)$

Inflection : $(0,0)$, $(1,-3)$

48/ $f(x) = x^2 + 8\ln|x+1|$

$\Rightarrow f'(x) = 2x + \frac{8}{x+1}$

$\Rightarrow f''(x) = 2 - \frac{8}{(x+1)^2}$

A/ $f''(x) = 0 \Rightarrow (x+1)^2 = 4 \Rightarrow x+1 = \pm 2$

$\Rightarrow x = 1$ or -3

B/ f'' undefined $\Rightarrow (x+1)^2 = 0 \Rightarrow x = -1$

$f'(3)$ exists $\qquad f'(-1)$ DNE $\qquad f'(1)$ exists $\qquad f''(x)$

$+ \qquad -3 \qquad - \qquad -1 \qquad - \qquad 1 \qquad +$

$f'(-4) > 0 \qquad f'(-2) < 0 \qquad f'(0) < 0 \qquad f''(2) > 0$

$\Rightarrow$ Concave Up : $(-\infty, -3)$ and $(1, \infty)$

Concave Down : $(-3, -1)$ and $(-1, 1)$

Inflection : $(-3, f(-3))$, $(1, f(1))$

59/ $f(x) = -x^2 - 10x - 25$

$\Rightarrow f'(x) = -2x - 10$

A/ $f'(x) = 0 \Rightarrow x = -5$

B/ f' continuous everywhere

$f''(x) = -2$

$\Rightarrow f''(-5) = 0$

and

$f''(-5) = -2 < 0$ $\qquad \Rightarrow f(-5)$ rel. max

63/ $f(x) = (x+3)^4 \Rightarrow f'(x) = 4(x+3)^3$

A/ $f'(x) = 0 \Rightarrow x = -3$

B/ f' continuous everywhere

$f''(x) = 12(x+3)^2$

$f''(-3) = 0 \Rightarrow 2^{nd}$ Derivative Test inconclusive

$- \qquad -3 \qquad + \qquad f'(x)$

$f'(-4) < 0 \qquad f'(0) > 0$

$\Rightarrow f(-3)$ rel. min

66/ $f(x) = x^{8/3} + x^{5/3} \Rightarrow f'(x) = \frac{8}{3}x^{5/3} + \frac{5}{3}x^{2/3}$

$= \frac{1}{3}x^{2/3}(8x+5)$

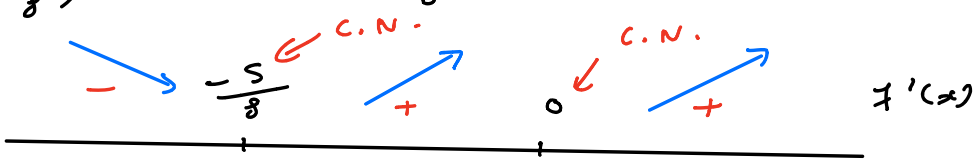
A/ $f'(x) = 0 \Rightarrow x = 0$ or $\frac{-5}{8}$

B/ f' continuous everywhere

$f''(x) = \frac{40}{9}x^{2/3} + \frac{10}{9}x^{-\frac{1}{3}}$

$f''(0)$ DNE $\Rightarrow$ must use 1^{st} derivative test

$f''(\frac{-5}{8}) > 0 \Rightarrow f(\frac{-5}{8})$ rel. min



$f'(-1) < 0$ $\quad f'(-\frac{1}{8}) > 0$ $\quad f'(1) > 0$

$\Rightarrow$ Only relative extrema is at $\frac{-5}{8}$.

76/ $u(M) = M^{1/2} \Rightarrow u'(M) = \frac{1}{2}M^{-\frac{1}{2}} \Rightarrow u''(M) = \frac{-1}{4}M^{-\frac{3}{2}}$

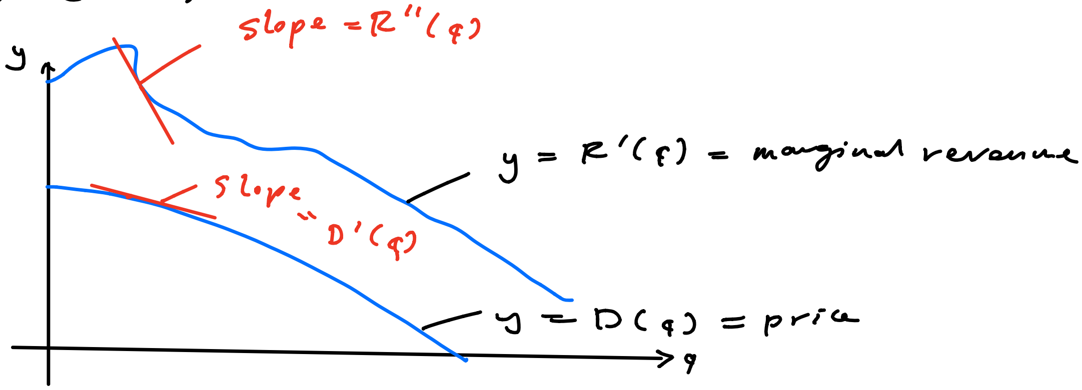
$$\Rightarrow \frac{-u''(M)}{u'(M)} = -\frac{\frac{-1}{4}M^{-\frac{3}{3}}}{\frac{1}{2}M^{-\frac{1}{2}}} = \frac{1}{2M}$$

$u(M) = M^{2/3} \Rightarrow u'(M) = \frac{2}{3}M^{-\frac{1}{3}} \Rightarrow u''(M) = \frac{-2}{9}M^{-\frac{4}{3}}$

$$\Rightarrow \frac{-u''(M)}{u'(M)} = -\frac{\frac{-2}{9}M^{-\frac{4}{3}}}{\frac{2}{3}M^{-3}} = \frac{1}{3M}$$

$\frac{1}{2M} > \frac{1}{3M} \Rightarrow u(M) = \sqrt{M}$ indicates a greater aversion to risk

77/ (Hard)



Marginal Revenue declines faster than price $\iff R''(q) < D'(q)$

$$R(q) = pq = D(q)\cdot q$$

$$\Rightarrow R'(q) = \frac{d}{dq}(D(q))\cdot q + D(q)\cdot\frac{dq}{dq} = D'(q)\,q + D(q)$$

$$R''(q) = D''(q)\cdot q + D'(q)\frac{d}{dq}(q) + D'(q)$$

$$= D''(q)\cdot q + D'(q) + D'(q)$$

$$R''(q) < D'(q) \iff D''(q)\cdot q + D'(q) + D'(q) < D'(q)$$

$$\iff D''(q)\,q + D'(q) < 0$$

79/ (Hard)

a)

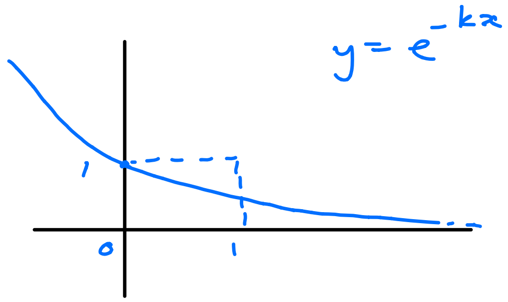
$$R(x) = \underbrace{Cx}_{u}\underbrace{(1-e^{-kx})}_{v} \qquad (C, k > 0)$$

$$u'(x) = C, \quad v'(x) = -e^{-kx}\cdot(-k) = ke^{-kx}$$

$$\Rightarrow R'(x) = u'(x)v(x) + u(x)v'(x)$$

$$= C(1-e^{-kx}) + Ckxe^{-k}$$

<u>Problem</u> : Cannot find critical numbers.



$\Rightarrow \quad 0 \le e^{-kx} \le 1$ on $[0,1]$

$\Rightarrow \quad 1 - e^{-kx} \geqslant 0$ on $[0,1]$

and

$x e^{-kx} \geqslant 0$ on $[0,1]$

$$\Rightarrow \quad R'(x) = C(1 - e^{-kx}) + Ckxe^{-kx} \geqslant 0 \text{ on } [0,1]$$

In fact $R'(x) > 0$ on $(0,1)$

$\Rightarrow$ $R(x)$ increasing on $[0,1]$ (continuous at end points)

b) $R''(x) = Cke^{-kx} + Cke^{-kx} - Ck^2xe^{-kx}$

A/ $R''(x) = 0 \Rightarrow (2Ck - Ck^2x)e^{-kx} = 0$

$\Rightarrow \quad 2Ck - Ck^2x = 0$

$\Rightarrow \quad x = \dfrac{2}{k}$

B/ R'' continuous everywhere.

$R''(x)$: $+$ ... $2/k$ ... $-$

$R''(\frac{1}{k}) > 0$ $\qquad$ $R''(\frac{3}{k}) < 0$

$\Rightarrow$ Concave up on $(0, \frac{2}{k})$.

83/ $K(t) = \dfrac{u(t)}{v(t)}$, $u(t) = 3t$, $v(t) = t^2 + 4$ $\quad (t \geqslant 0)$

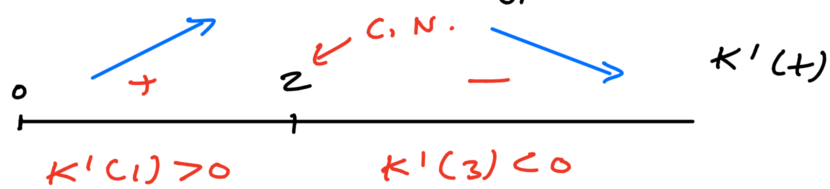
$\Rightarrow u'(t) = 3$, $v'(t) = 2t$

$$\Rightarrow \quad K'(t) = \frac{u'(t)v(t) - u(t)v'(t)}{(v(t))^2}$$

$$= \frac{3(t^2+4) - 3t \cdot 2t}{(t^2+4)^2} = \frac{12 - 3t^2}{(t^2+4)^2}$$

A/ $K'(t) = 0 \Rightarrow t = \pm 2$

B/ $t^2 + 4 > 0 \Rightarrow$ No type B/ points



a) Concentration is max at $t = 2$

b) $K(2) = \frac{3 \cdot 2}{2^2 + 4} = \frac{3}{4}$ %.

§5.4

7/ $f(x) = x^4 - 24x^2 + 80$

Domain : $(-\infty, \infty)$

y-intercept : $(0, 80)$

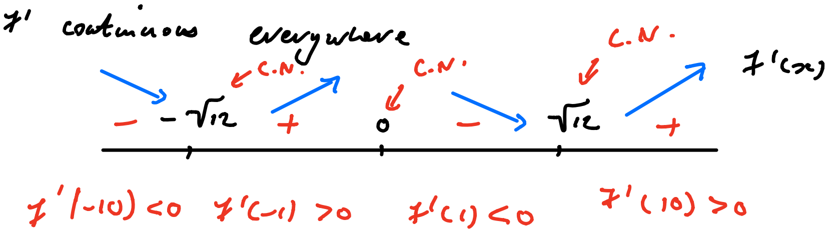
No horizontal /vertical asymptotes $\left(\lim_{x \to \pm\infty} f(x) = \infty \right)$

Even

$f'(x) = 4x^3 - 48x = 4x(x^2 - 12)$

A/ $f'(x) = 0 \Rightarrow x = 0, \pm\sqrt{12}$

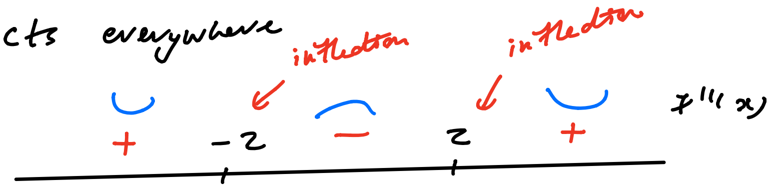
B/ f' continuous everywhere



$f''(x) = 12x^2 - 48 = 12(x^2 - 4)$

4/ $f''(x) = 0 \Rightarrow x = \pm 2$

5/ f'' cts everywhere

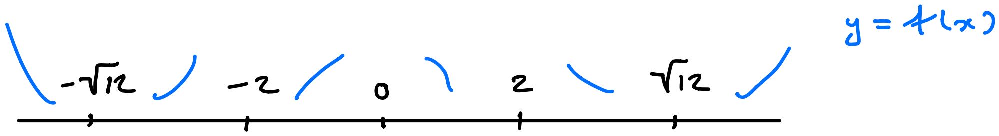


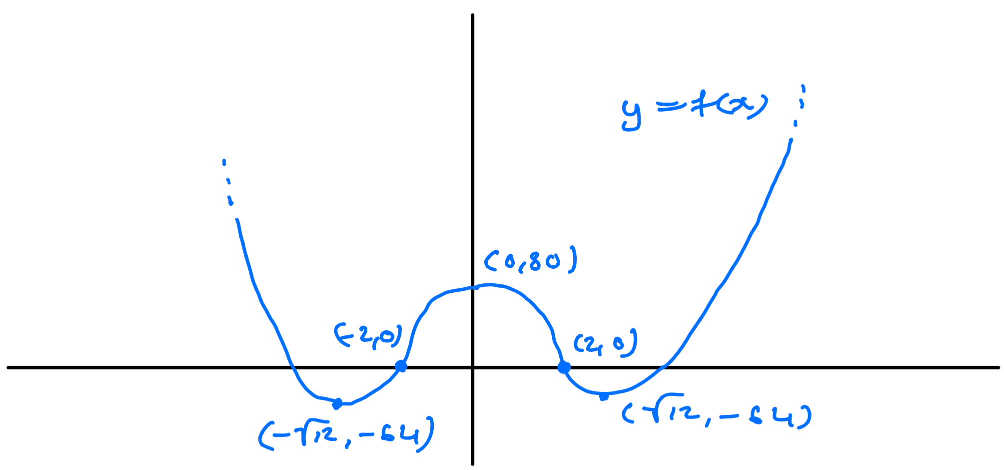
$f''(-10) > 0$ $\quad$ $f'(0) < 0$ $\quad$ $f''(10) > 0$

$f(0) = 80$

$f(\sqrt{12}) = f(-\sqrt{12}) = -64$

$f(2) = f(-2) = 0$





11/ (Don't worry about slant asymptote in this question)

$f(x) = 2x + \frac{10}{x}$

Domain : $x \neq 0$

No y-intercept

$f(x) = 0 \Rightarrow 2x + \frac{10}{x} = 0 \Rightarrow 2x = \frac{-10}{x} \Rightarrow x^2 = -5$

(No solutions)

No x-intercept

$\lim_{x \to \infty} 2x + \frac{10}{x} = \infty$, $\lim_{x \to -\infty} 2x + \frac{10}{x} = -\infty$

$\Rightarrow$ no horizontal asymptote

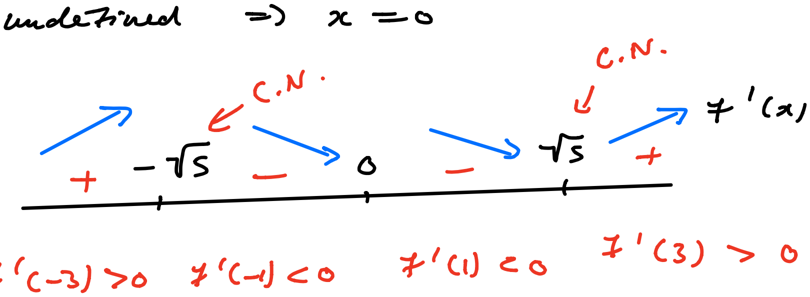
Vertical asymptote at $x = 0$

$f(x)$ odd (ie $f(-x) = -f(x)$)

$f'(x) = 2 - \frac{10}{x^2}$

A/ $f'(x) = 0 \Rightarrow 2 - \frac{10}{x^2} = 0 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

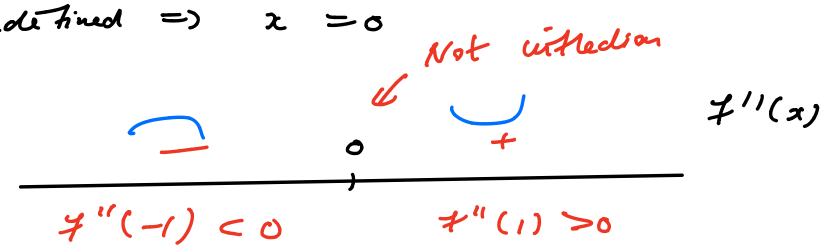
B/ f' undefined $\Rightarrow x = 0$



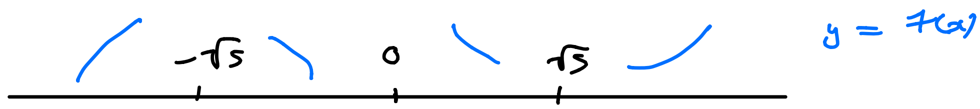
$f''(x) = \frac{20}{x^3}$

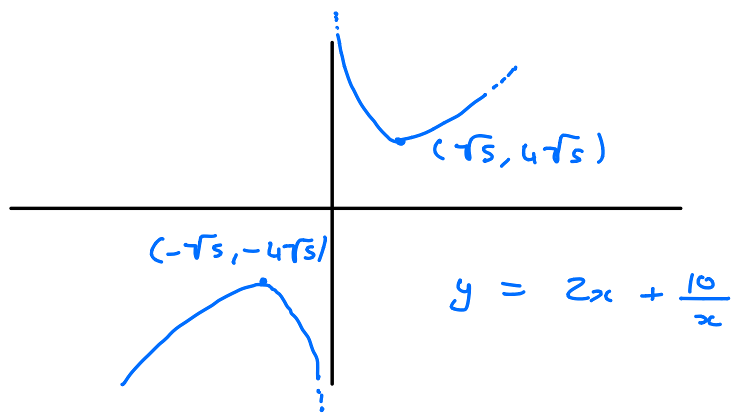
A/ $f''(x) = 0 \Rightarrow \frac{20}{x^3} = 0 \Rightarrow$ No solutions

B/ f'' undefined $\Rightarrow x = 0$



$f(\sqrt{5}) = 4\sqrt{5}$
$f(-\sqrt{5}) = -4\sqrt{5}$





20/ $f(x) = \dfrac{-2x}{x^2-4}$

<u>Domain</u> : $x \neq \pm 2$

y-intercept $= x$-intercept $= (0, 0)$

$\lim_{x \to \pm\infty} f(x) = 0 \Rightarrow y = 0$ horizontal asymptote

$x = 2$, $x = -2$ vertical asymptotes.

$f(x) = \dfrac{u(x)}{v(x)}$, $u(x) = -2x$, $v(x) = x^2 - 4$

$\Rightarrow u'(x) = -2$, $v'(x) = 2x$

$$\Rightarrow f'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{(v(x))^2} = \frac{-2(x^2-4) - (-2x)(2x)}{(x^2-4)^2}$$

$$= \frac{2x^2+8}{(x^2-4)^2}$$

Note $2x^2+8 > 0$ and $(x^2-4)^2 \geqslant 0 \Rightarrow f'(x) > 0$ whenever it is defined. Hence f increasing on every interval on which it is defined.

$$f'(x) = \frac{a(x)}{b(x)}, \quad a(x) = 2x^2+8, \quad b(x) = (x^2-4)^2$$

$$\Rightarrow \quad a'(x) = 4x, \quad b'(x) = 2(x^2-4)\cdot 2x$$

$$\Rightarrow \quad f''(x) = \frac{a'(x)\,b(x) - a(x)\,b'(x)}{(b(x))^2}$$

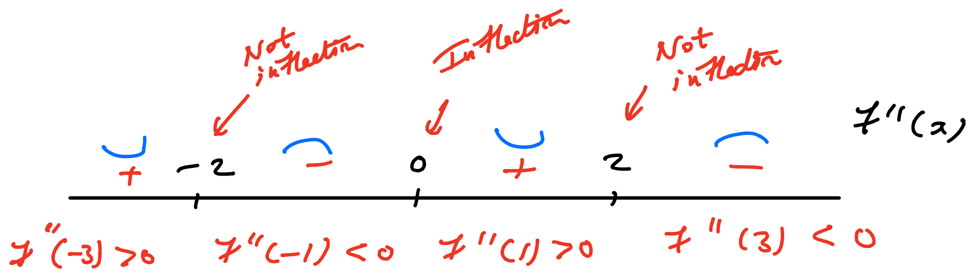
$$= \frac{4x\,(x^2-4)^2 - (2x^2+8)\cdot 2(x^2-4)\cdot 2x}{(x^2-4)^4}$$

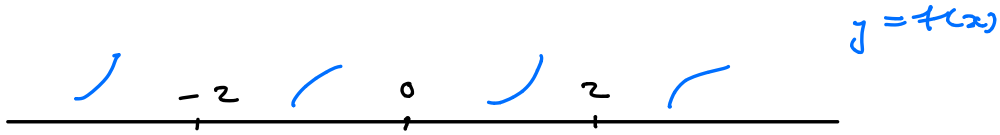
$$= \frac{4x(x^2-4) - 4x\,(2x^2+8)}{(x^2-4)^3}$$

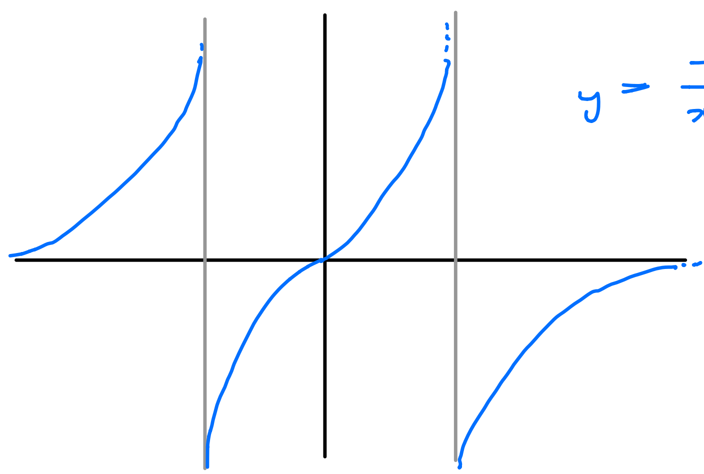
$$= \frac{-4x^3-48x}{(x^2-4)^3} = \frac{-4x(x^2+12)}{(x^2-4)^3}$$

A/ $f''(x) = 0 \Rightarrow -4x(x^2+12) = 0 \Rightarrow x = 0$

B/ f'' undefined $\Rightarrow x = \pm 2$







$$y = \frac{-2x}{x^2-4}$$

22/ $f(x) = x - \ln|x|$

Domain : $x \neq 0$

No y-intercept. $f(x) = 0$ to hard to solve.

$\lim_{x \to \pm\infty} f(x)$ to difficult. Not sure about horizontal asymptote

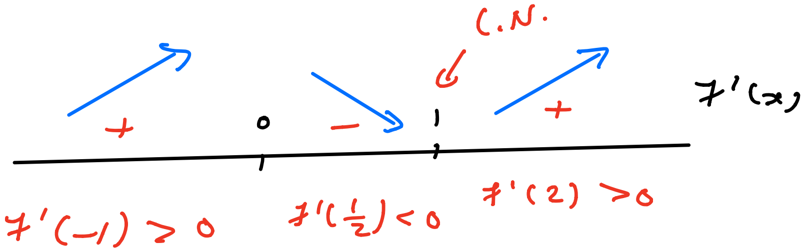
$x = 0$ vertical asymptote

Neither odd / even

$f'(x) = 1 - \frac{1}{x}$

A/ $f'(x) = 0 \Rightarrow x = 1$

B/ f' undefined $\Rightarrow x = 0$



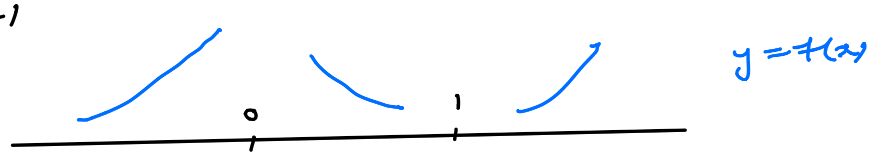
$f''(x) = \frac{1}{x^2} \Rightarrow f''(x) > 0$ when defined

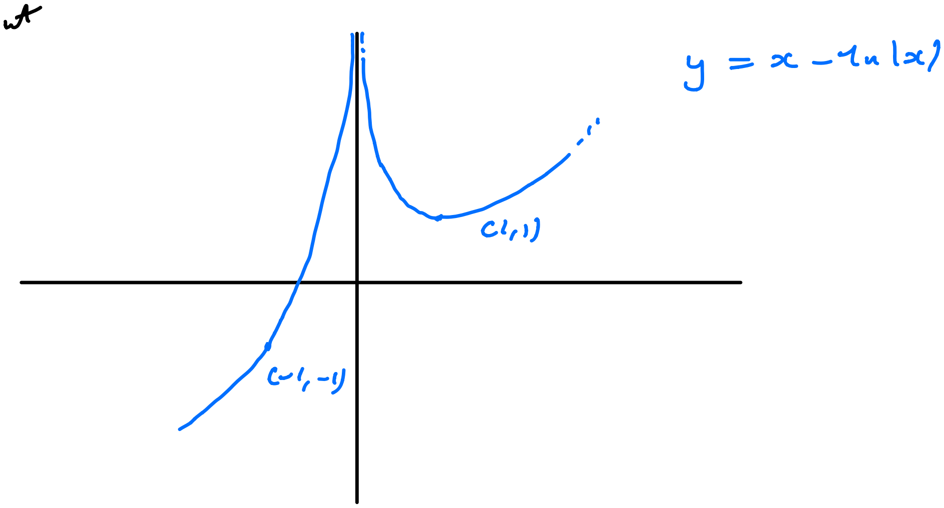
$\Rightarrow f$ always concave up

$f(1) = 1$

$f(-1) = -1$

just to give another concrete point on graph





23/ $f(x) = \dfrac{\ln(x)}{x}$

Domain : $(0, \infty)$

$f(x) = 0 \Rightarrow x = 1 \Rightarrow (1, 0)$ x-intercept

$\lim_{x \to \infty} f(x) = ??$ Not sure about horizontal asymptotes

$x = 0$ vertical asymptote

$f(x) = \dfrac{u(x)}{v(x)}$, $u(x) = \ln(x)$, $v(x) = x$

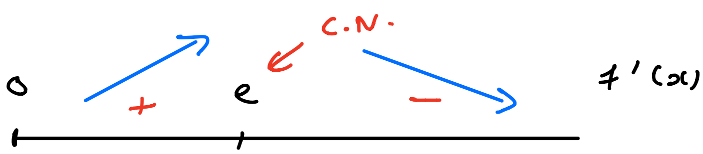
$\Rightarrow u'(x) = \dfrac{1}{x}$, $v'(x) = 1$

$$\Rightarrow f'(x) = \frac{u'(x)\, v(x) - u(x)\, v'(x)}{(v(x))^2}$$

$$= \frac{1 - \ln(x)}{x^2}$$

A/ $f'(x) = 0 \Rightarrow 1 - \ln(x) = 0 \Rightarrow \ln(x) = 1 \Rightarrow x = e$

B/ f' undefined $\Rightarrow x \leq 0$



$f'(e^{\frac{1}{2}}) = \dfrac{1 - \frac{1}{2}}{e} > 0$ $f'(e^2) = \dfrac{1-2}{e^4} < 0$

$f'(x) = \dfrac{a(x)}{b(x)}$, $a(x) = 1 - \ln(x)$, $b(x) = x^2$

$\Rightarrow a'(x) = \dfrac{-1}{x}$, $b'(x) = 2x$

$$\Rightarrow f''(x) = \frac{a'(x)\, b(x) - a(x)\, b'(x)}{(b(x))^2}$$

$$= \frac{\frac{-1}{x}x^2 - (1-\ln(x))\cdot 2x}{x^4}$$

$$= \frac{-1 - (1-\ln(x))2}{x^3}$$

$$= \frac{2\ln(x) - 3}{x^3}$$

A/ $f''(x) = 0 \Rightarrow 2\ln(x) - 3 = 0 \Rightarrow x = e^{3/2}$

B/ f'' undefined $\Rightarrow x \leq 0$

Inflection

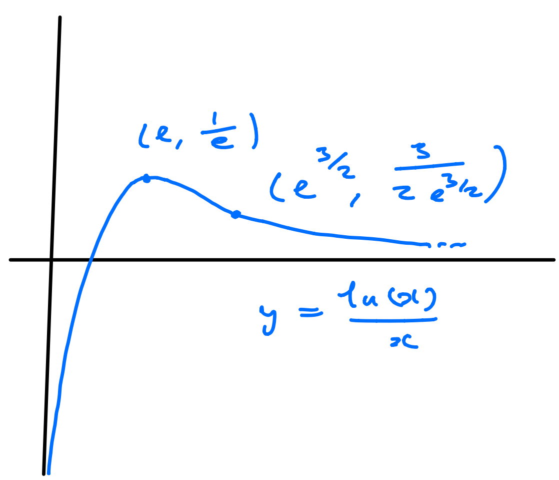
0	$-$	$e^{3/2}$	$+$	$f''(x)$

$f''(e) = \frac{2-3}{e^3} < 0 \quad f''(e^2) = \frac{4-3}{e^6} > 0$

0	e	$e^{3/2}$	$y = f(x)$

$f(e) = \frac{1}{e}$

$f(e^{3/2}) = \frac{3}{2e^{3/2}}$



28/ $f(x) = e^x + e^{-x}$

Domain : $(-\infty, \infty)$

$f(0) = 2 \Rightarrow (0,2)$ y-intercept

$f(x) > 0 \Rightarrow$ no x-intercept

$\lim\limits_{x \to \pm\infty} f(x) = \infty \Rightarrow$ No horizontal asymptotes

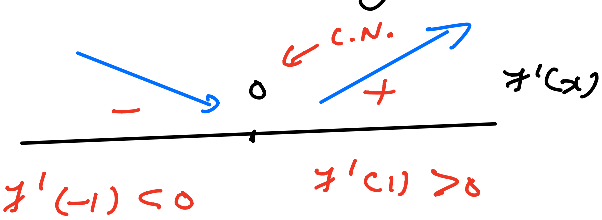
No vertical asymptotes.

Even

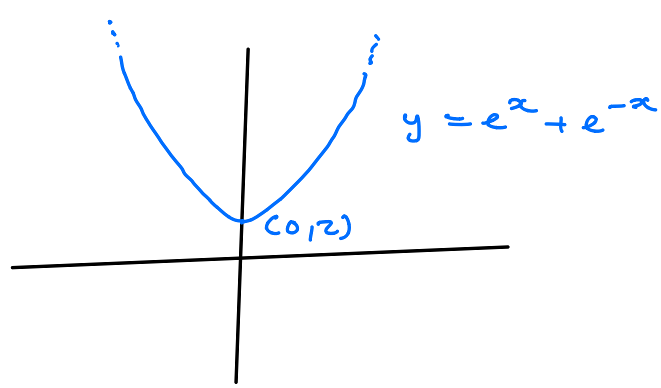
$f'(x) = e^x - e^{-x}$

A/ $f'(x) = 0 \Rightarrow e^x = e^{-x} \Rightarrow e^{2x} = 1 \Rightarrow 2x = 0$
$\Rightarrow x = 0$

B/ f' continuous everywhere



$f''(x) = e^x + e^{-x} > 0 \Rightarrow f$ always concave up.



24/ $f(x) = x^{2/3} - x^{5/3}$

Domain : $(-\infty, \infty)$

$f(0) = 0 \Rightarrow (0,0) = y$-intercept

$f(x) = 0 \Rightarrow x^{2/3} - x^{5/3} = 0 \Rightarrow x^{2/3}(1-x) = 0$

$\Rightarrow x = 0$ or 1

$\Rightarrow (0,0)$ and $(1,0)$ are x-intercepts

$\lim_{x \to \pm\infty} f(x) = ??$ Don't know about horizontal asymptotes

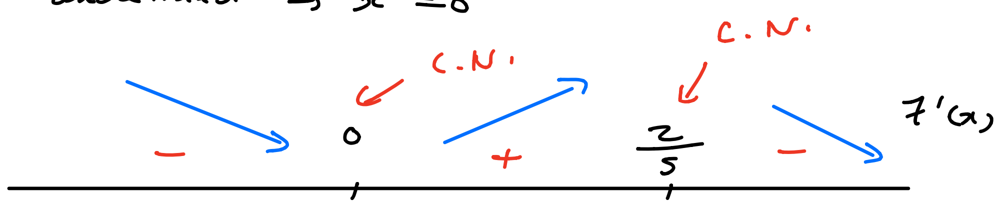
No vertical asymptotes

Neither odd or even.

$f'(x) = \frac{2}{3} x^{-\frac{1}{3}} - \frac{5}{3} x^{2/3} = \frac{2}{3} \frac{1}{x^{1/3}} - \frac{5}{3} x^{2/3}$

A/ $f'(x) = 0 \Rightarrow \frac{2}{3} \frac{1}{x^{1/3}} = \frac{5}{3} x^{2/3} \Rightarrow x = \frac{2}{5}$

B/ f' undefined $\Rightarrow x = 0$



$f'(-1) < 0$

$f'(1) < 0$

(Recall : $(\frac{1}{8})^{\frac{1}{3}} = \frac{1}{2}$) $f(\frac{1}{8}) = \frac{2}{3} \cdot \frac{1}{(\frac{1}{2})} - \frac{5}{2} \cdot (\frac{1}{2})^2 > 0$

$f''(x) = \frac{-2}{9} x^{-\frac{4}{3}} - \frac{10}{9} x^{-\frac{1}{3}}$

$= \frac{-2}{9} \cdot \frac{1}{x^{4/3}} - \frac{10}{9} \frac{1}{x^{1/3}}$

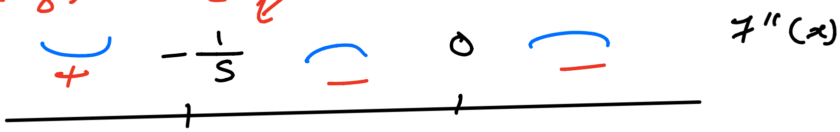
A/ $f''(x) = 0 \Rightarrow \frac{-2}{9} \cdot \frac{1}{x^{4/3}} = \frac{10}{9} \cdot \frac{1}{x^{1/3}}$

$$\Rightarrow \quad x = \frac{-2}{10} = \frac{-1}{5}$$

B/ f'' undefined $\Rightarrow$ $x = 0$

Recall: $\left(\frac{-1}{8}\right)^{1/3} = \frac{-1}{2}$

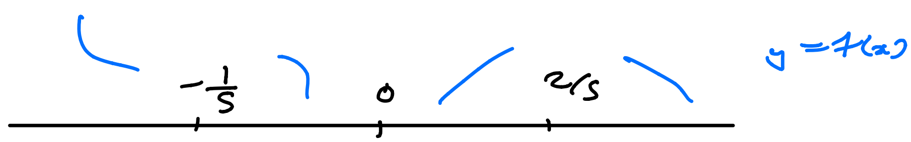
Inflection



$f''(-1) > 0$

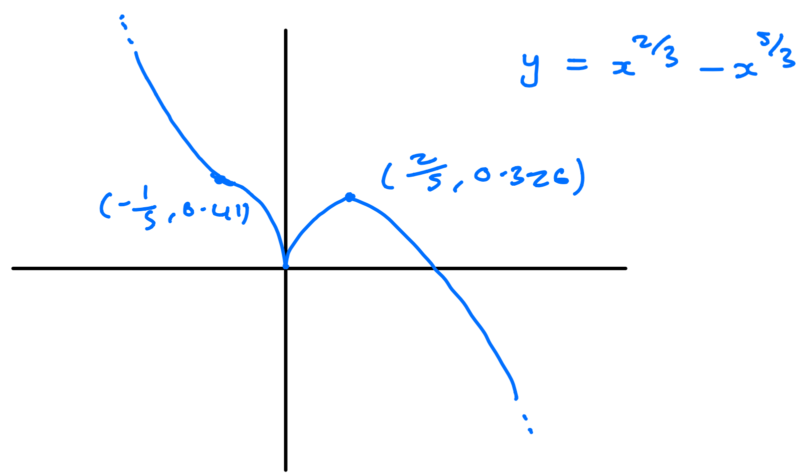
$f''(1) < 0$

$$f''\left(\frac{-1}{8}\right) = \frac{-2}{9} \cdot \frac{1}{\left(\frac{-1}{2}\right)^4} - \frac{10}{9} \cdot \frac{1}{\left(-\frac{1}{2}\right)} < 0$$



$f\left(\frac{-1}{5}\right) \approx 0.41$

$f\left(\frac{2}{5}\right) \approx 0.326$



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