# Homework 6 Solutions

Throughout these solutions we will freely use these formulae:

$$\frac{d}{dx}\left(a^{g(x)}\right) = \ln(a)\, a^{g(x)} g'(x)$$

$$\frac{d}{dx} \log_a |g(x)| = \frac{g'(x)}{\ln(a)\, g(x)}$$

## §4.5

3/ $\frac{d}{dx} \ln(8-3x) = \frac{-3}{8-3x}$

5/ $\frac{d}{dx} \ln|4x^2 - 9x| = \frac{8x-9}{4x^2-9x}$

7/ $\frac{d}{dx}\left(\ln\left(\sqrt{x+5}\right)\right) = \frac{d}{dx}\left(\frac{1}{2}\ln(x+5)\right) = \frac{1}{2} \cdot \frac{1}{x+5}$

12/ $y = uv$, $u = 3x+7$, $v = \ln(2x-1) \Rightarrow \frac{du}{dx} = 3$, $\frac{dv}{dx} = \frac{2}{2x-1}$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} v + u\frac{dv}{dx} = 3\ln(2x-1) + (3x+7) \cdot \frac{2}{(2x-1)}$

14/ $y = uv$, $u = x$, $v = \ln|2-x^2| \Rightarrow \frac{du}{dx} = 1$, $\frac{dv}{dx} = \frac{-2x}{2-x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} v + u\frac{dv}{dx} = 1 \cdot \ln|2-x^2| + x \cdot \frac{-2x}{2-x^2}$

17/ $y = \frac{u}{v}$, $u = \ln(x)$, $v = 4x+7 \Rightarrow \frac{du}{dx} = \frac{1}{x}$, $\frac{dv}{dx} = 4$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{du}{dx} v - u\frac{dv}{dx}}{v^2} = \frac{\frac{1}{x}\cdot(4x+7) - \ln(x)\cdot 4}{(4x+7)^2}$$

41/ $y = f(x) = uv$, $u = e^{\sqrt{x}}$, $v = \ln(\sqrt{x}+5) \Rightarrow \frac{du}{dx} = e^{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}}$

$\frac{dv}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{x}+5} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} v + u\frac{dv}{dx}$

$$= e^{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}} \cdot \ln(\sqrt{x}+5) + e^{\sqrt{x}} \cdot \frac{\frac{1}{2}x^{-\frac{1}{2}}}{\sqrt{x}+5}$$

43/ $y = \frac{u}{v}$, $u = \ln(t^2+1) + t$, $v = \ln(t^2+1) + 1$

$$\Rightarrow \frac{du}{dt} = \frac{2t}{t^2+1} + 1, \quad \frac{dv}{dt} = \frac{2t}{t^2+1}$$

$$\Rightarrow \frac{dy}{dt} = \frac{\frac{du}{dt} v - u \frac{dv}{dt}}{v^2}$$

$$= \frac{\left(\frac{2t}{t^2+1} + 1\right)\left(\ln(t^2+1) + 1\right) - \left(\ln(t^2+1) + t\right)\left(\frac{2t}{t^2+1}\right)}{\left(\ln(t^2+1) + 1\right)^2}$$

46/ $$\frac{d}{dx} \ln|ax| = \frac{\frac{d}{dx}(ax)}{ax} = \frac{a}{ax} = \frac{1}{x} = \ln(|x|)$$

57/ $$p = D(q) = 100 + \frac{50}{\ln(q)}$$

$$\Rightarrow R(q) = pq = 100q + \frac{50q}{\ln(q)}$$

a) $$\frac{dR}{dq} = 100 + 50 \frac{d}{dq}\left(\frac{q}{\ln(q)}\right)$$

$$= 100 + 50 \cdot \frac{\frac{d}{dq}(q)\ln(q) - q\frac{d}{dq}(\ln(q))}{(\ln(q))^2}$$

$$= 100 + \frac{50\ln(q) - 1}{(\ln(q))^2}$$

b) $$R'(q) \approx R(q+1) - R(q)$$

Let $q = 8 \Rightarrow R'(8) \approx R(9) - R(8)$

$$\Rightarrow R(9) \approx R'(8) + R(8) \approx \$112.48$$

c) The manager could use this info to decide whether it is sensible to sell addition units.

60/ a) $R'(x) = \frac{d}{dx} 30 \ln(2x+1) = 30 \cdot \frac{2}{2x+1}$

b) $P(x) = R(x) - C(x) = 30 \ln(2x+1) - \frac{x}{2}$

c) $P'(x) = 30 \cdot \frac{2}{2x+1} - \frac{1}{2}$

$P'(60) = \frac{60}{121} - \frac{1}{2}$

d) If they sell an addition unit they will make approximately $\frac{60}{121} - \frac{1}{2}$ extra. That's not very much.

66/ $P'(t) = \frac{-5.74}{t}$

a) $P(5) \approx 21.28\%$ $\quad P'(5) \approx -1.158\%$ per year

b) $P(25) \approx 11.96\%$, $P'(25) \approx -0.2316\%$ per year

c) $P(49) \approx 8.67\%$, $P'(49) \approx -0.1182\%$ per year

d) $\lim_{t \to \infty} P'(t) = 0$

§5.1

1/ a) $(1, \infty)$ ; b) $(-\infty, 1)$

7/ a) $(-7, -4)$, $(-2, \infty)$ ; b) $(-\infty, -7)$, $(-4, -2)$

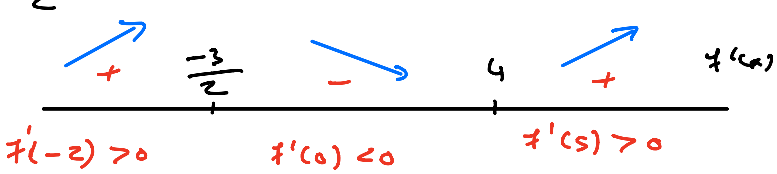
17/ $f(x) = 4x^3 - 15x^2 - 72x + 5$

$\Rightarrow f'(x) = 12x^2 - 30x - 72 = 6(2x^2 - 5x - 12) = 6(2x+3)(x-4)$

A/ $f'(x) = 0 \Rightarrow 6(2x+3)(x-4) = 0 \Rightarrow x = \frac{-3}{2}, 4$

B/ $f'$ continuous everywhere

$\Rightarrow \frac{-3}{2}, 4$ critical numbers



$f$ increasing on $(-\infty, \frac{-3}{2})$ and $(4, \infty)$
$f$ decreasing on $(\frac{-3}{2}, 4)$.

23/ $f(x) = \dfrac{x+2}{x+1} = \dfrac{u}{v}$ , $u = x+2$ , $v = x+1 \Rightarrow \dfrac{du}{dx} = \dfrac{dv}{dx} = 1$

$$\Rightarrow f'(x) = \frac{\frac{du}{dx} v - u \frac{dv}{dx}}{v^2} = \frac{(x+1) - (x+2)}{(x+2)^2} = \frac{-1}{(x+1)^2}$$

A/ $f'(x) = 0 \Rightarrow \dfrac{-1}{(x+1)^2} = 0$ (No solution)

B/ $f'$ undefined $\Rightarrow (x+1)^2 = 0 \Rightarrow x = -1$

$f(-1)$ DNE $\Rightarrow$ No critical numbers

$f'(x)$: $-$ | $-1$ | $-$

$f'(-2) < 0$ $\quad$ $f'(0) < 0$

$\Rightarrow f$ decreasing on $(-\infty, -1)$ and $(-1, \infty)$.

31/ $f = uv$ , $u = x$ , $v = e^{-3x} \Rightarrow \dfrac{du}{dx} = 1$ , $\dfrac{dv}{dx} = e^{-3x} \cdot (-3)$

$$\Rightarrow f'(x) = 1 \cdot e^{-3x} + x \cdot e^{-3x} \cdot (-3)$$
$$= \underbrace{e^{-3x}}_{>0} (1 - 3x)$$

A/ $f'(x) = 0 \Rightarrow (1-3x) = 0 \Rightarrow x = \frac{1}{3}$

B/ $f'$ continuous everywhere

$\Rightarrow \frac{1}{3}$ only critical number

$f'(x)$: $+$ | $\frac{1}{3}$ | $-$

$f'(0) > 0$ $\quad$ $f'(1) < 0$

$\Rightarrow$ $f$ increasing on $(-\infty, \frac{1}{3})$
$f$ decreasing on $(\frac{1}{3}, \infty)$

34/ $f = uv$, $u = x$, $v = 2^{(-x^2)}$ $\Rightarrow \frac{du}{dx} = 1$, $\frac{dv}{dx} = \ln(2)\, 2^{-x^2} \cdot (-2x)$

$$\Rightarrow f'(x) = \frac{du}{dx} v + u \frac{dv}{dx}$$
$$= 1 \cdot 2^{(-x^2)} + x \ln(2)\, 2^{(-x^2)} \cdot (-2x)$$
$$= \underbrace{2^{(-x^2)}}_{>0} \left(1 - 2\ln(2)\, x^2\right)$$

A/ $f'(x) = 0 \Rightarrow 1 - 2\ln(2) x^2 = 0 \Rightarrow x = \pm \sqrt{\frac{1}{2\ln(2)}}$

B/ $f'$ continuous everywhere

$\Rightarrow \pm \sqrt{\frac{1}{2\ln(2)}}$ critical numbers

$-$ $\quad -\sqrt{\frac{1}{2\ln(2)}} \quad$ $+$ $\quad \sqrt{\frac{1}{2\ln(2)}} \quad$ $-$ $\quad f'(x)$

$f'(-10) < 0 \qquad f'(0) > 0 \qquad f'(10) < 0$

$\Rightarrow f$ increasing on $\left(-\sqrt{\frac{1}{2\ln(2)}}, \sqrt{\frac{1}{2\ln(2)}}\right)$

$f$ decreasing on $\left(-\infty, -\sqrt{\frac{1}{2\ln(2)}}\right)$ and $\left(\sqrt{\frac{1}{2\ln(2)}}, \infty\right)$.

36/ $f(x) = x^{1/3} + x^{4/3} \Rightarrow f'(x) = \frac{1}{3} x^{-2/3} + \frac{4}{3} x^{1/3} = \frac{1}{3} \cdot \frac{1}{x^{2/3}} + \frac{4}{3} x^{1/3}$

A/ $f'(x) = 0 \Rightarrow \frac{1}{3} \cdot \frac{1}{x^{2/3}} + \frac{4}{3} x^{1/3} = 0$

$$\Rightarrow \frac{1}{3} \cdot \frac{1}{x^{2/3}} = \frac{-4}{3} x^{1/3}$$
$$\Rightarrow \frac{-1}{4} = x$$

B/ $f'$ undefined $\Rightarrow x^{2/3} = 0 \Rightarrow x = 0$

$f$ continuous everywhere $\Rightarrow 0, \frac{-1}{4}$ critical numbers

$-$ $\quad -\frac{1}{4} \quad$ $+$ $\quad 0 \quad$ $+$ $\quad f'(x)$

$f'(-1) = \frac{1}{3} \cdot \frac{1}{(-1)^2} - \frac{4}{3} < 0 \qquad f'(-\frac{1}{8}) = \frac{1}{3} \cdot \frac{1}{(-\frac{1}{2})^2} + \frac{4}{3}(-\frac{1}{2}) \qquad f'(1) > 0$

Recall: $\left(\frac{-1}{8}\right)^{\frac{1}{3}} = \frac{-1}{2}$ $\qquad = \frac{4}{3} - \frac{4}{6} > 0$

$\Rightarrow$ $f$ increasing on $\left(-\frac{1}{4}, 0\right)$ and $(0, \infty)$

$\left(\text{actually on } \left(\frac{-1}{4}, \infty\right)\right)$

$f$ decreasing on $\left(-\infty, \frac{-1}{4}\right)$.

45/ $$H(r) = \frac{300}{1+0.03r^2} = 300 \cdot (1+0.03r^2)^{-1}$$

$$\Rightarrow \quad H = 300u^{-1}, \quad u = 1+0.03r^2$$

$$\Rightarrow \quad \frac{dH}{dr} = \frac{dH}{du} \cdot \frac{du}{dr}$$

$$= -300u^{-2} \cdot (0.06)r$$

$$= \frac{-18r}{(1+0.03r^2)^2} \qquad (r \geqslant 0)$$

mortgage rate

A/ $H'(r) = 0 \Rightarrow -18r = 0 \Rightarrow r = 0$

B/ $H'$ undefined $\Rightarrow (1+0.03r^2)^2 = 0$

$\Rightarrow 1+0.03r^2 = 0$

$\Rightarrow r^2 = \frac{-1}{0.03} < 0$

(No solutions

$0$ — $H'(r)$

$H'(1) < 0$

a) $H(r)$ is never increasing on $[0, \infty)$

b) $H(r)$ is decreasing on $(0, \infty)$.

48/ $$P(x) = -(x-4)e^x - 4$$

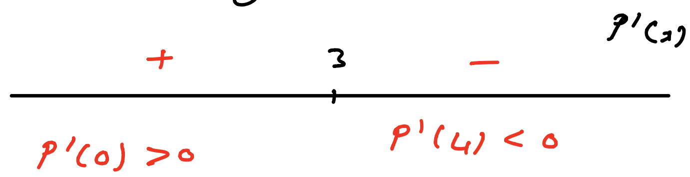
$$\Rightarrow \quad P'(x) = -\left(\frac{d}{dx}(x-4)\,e^x + (x-4)\frac{d}{dx}e^x\right)$$

$$= -\left(e^x + (x-4)e^x\right)$$

$$= -e^x\left(1 + (x-4)\right) = -e^x(x-3)$$

A/ $P'(x) = 0 \Rightarrow x-3 = 0 \Rightarrow x = 3$

3/ $p'$ continuous everywhere




$\Rightarrow$ $p$ increasing on $(0,3)$

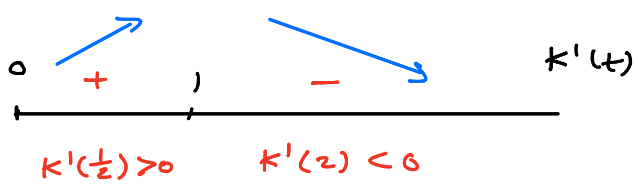
$p$ decreasing on $(3, 3.9)$

53/ $K(t) = \dfrac{u(t)}{v(t)}$ , $u(t) = 5t$ , $v(t) = t^2+1$ $\Rightarrow u'(t) = 5$ , $v'(t) = 2t$

$$\Rightarrow K'(t) = \frac{u'(t)v(t) - u(t)v'(t)}{[v(t)]^2} = \frac{5(t^2+1) - 5t \cdot 2t}{(t^2+1)^2} = \frac{5-5t^2}{(t^2+1)^2}$$

A/ $K'(t) = 0 \Rightarrow 5 - 5t^2 = 0 \Rightarrow t = \pm 1$ (only care about $t \geq 0$)

B/ $K'$ continuous everywhere $(t^2+1 > 0)$




$\Rightarrow$ $K$ increasing on $(0,1)$

$K$ decreasing on $(1, \infty)$

60/ a) About $(1945, 1965)$ and $(1970, 1973)$

b) About $(1987, 2010)$

§5.2

3/ Relative max of 3 at $-2$

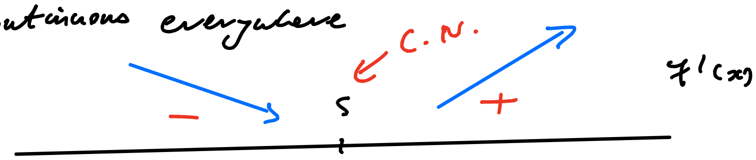
8/ Relative max of 4 at 0 ; Relative min of 0 at $-3$ and 3

13/ $f(x) = x^2 - 10x + 33$

$\Rightarrow f'(x) = 2x - 10$

A/ $f'(x) = 0 \Rightarrow x = 5$

B/ $f'$ continuous everywhere




$f'(4) = -2 < 0 \quad f'(6) = 2 > 0$

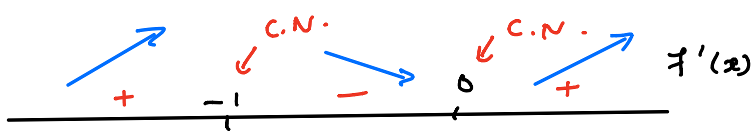
$\Rightarrow f(5) = 8$ is rel. min.

23/ $f(x) = 2x + 3x^{2/3}$

$\Rightarrow f'(x) = 2 + 2x^{-\frac{1}{3}} = 2 + \dfrac{2}{x^{1/3}}$

A/ $f'(x) = 0 \Rightarrow 2 + \dfrac{2}{x^{1/3}} = 0 \Rightarrow x^{1/3} = -1 \Rightarrow x = (-1)^3 = -1$

B/ $f'$ undefined $\Rightarrow x^{1/3} = 0 \Rightarrow x = 0$




$f'(-8) = 2 + \frac{2}{-2} = 1 > 0 \quad f'(1) = 4 > 0$

$f'\left(\frac{-1}{8}\right) = 2 + \dfrac{2}{\left(\frac{-1}{2}\right)} = 2 - 4 = -2 < 0$

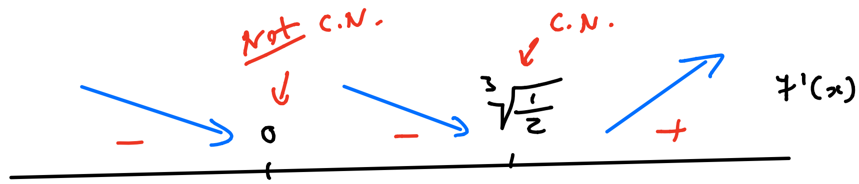
$\Rightarrow f(-1) = 1$ rel. max

$f(0) = 0$ rel. min

26/ $f(x) = x^2 + \dfrac{1}{x} \Rightarrow f'(x) = 2x - \dfrac{1}{x^2}$

A/ $f'(x) = 0 \Rightarrow 2x - \frac{1}{x^2} = 0 \Rightarrow x^3 = \frac{1}{2} \Rightarrow \sqrt[3]{\frac{1}{2}}$

B/ $f'$ undefined $\Rightarrow x^2 = 0 \Rightarrow x = 0$




$f'(-1) = -3 < 0$ $\qquad$ $f'(1) = 1 > 0$

$f'(\frac{1}{2}) = 2 \cdot \frac{1}{2} - \frac{1}{(\frac{1}{4})} = -3 < 0$

$\Rightarrow \quad f(\sqrt[3]{\tfrac{1}{2}}) = (\tfrac{1}{2})^{2/3} - \dfrac{1}{(\sqrt[3]{\frac{1}{2}})}$ rel. min

Ex/ $f(x) = \dfrac{u(x)}{v(x)}$, $u(x) = x^2 - 2x + 1$, $v(x) = x - 3$

$\Rightarrow u'(x) = 2x - 2$, $v'(x) = 1$

$$\Rightarrow f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$$

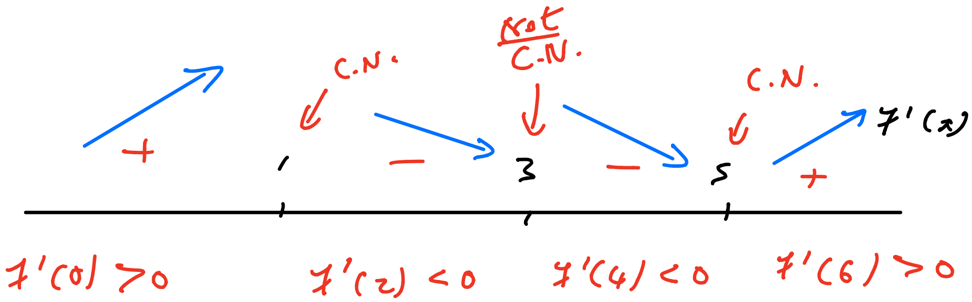
$$= \frac{(2x-2)(x-3) - (x^2 - 2x + 1)}{(x-3)^2}$$

$$= \frac{2x^2 - 6x - \cancel{2x} + 6 - x^2 + \cancel{2x} - 1}{(x-3)^2}$$

$$= \frac{x^2 - 6x + 5}{(x-3)^2} = \frac{(x-1)(x-5)}{(x-3)^2}$$

A/ $f'(x) = 0 \Rightarrow (x-1)(x-5) = 0 \Rightarrow x = 1, 5$

B/ $f'$ undefined $\Rightarrow (x-3)^2 = 0 \Rightarrow x = 3$




$\Rightarrow \quad f(1) = 0 \quad$ rel. max

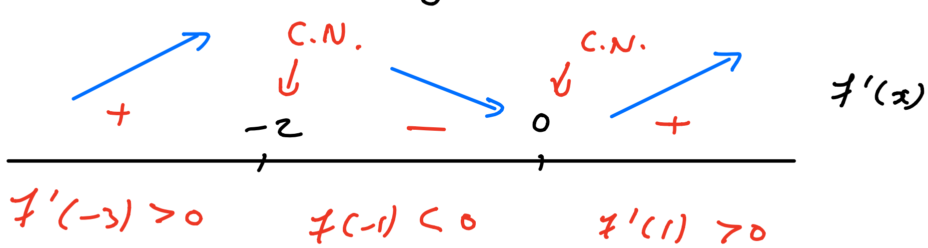
$\quad\;\; f(5) = 8 \quad$ rel. min

24/ $\quad f(x) = x^2 e^x - 3 \quad \Rightarrow f'(x) = \frac{d}{dx}(x^2)\, e^x + x^2 \frac{d}{dx}(e^x) - \frac{d}{dx}(3)$

$\Rightarrow \quad f'(x) = \underbrace{e^x}_{>0}(2x + x^2) = e^x \cdot x\,(2+x)$

A/ $\quad f'(x) = 0 \Rightarrow \quad x(2+x) = 0 \quad \Rightarrow x = 0, -2$

B/ $\quad f'$ continuous everywhere




$f(-2) = 4e^{-2} - 3 \quad$ rel. max.

$f(0) = -3 \quad$ rel. min.

32/ $\quad f(x) = \dfrac{u(x)}{v(x)}, \quad u(x) = x^2, \quad v(x) = \ln(x)$

$\Rightarrow \quad u'(x) = 2x, \quad v'(x) = \frac{1}{x}$

$\Rightarrow \quad f'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} = \dfrac{2x\ln(x) - x^2 \cdot \frac{1}{x}}{(\ln(x))^2}$

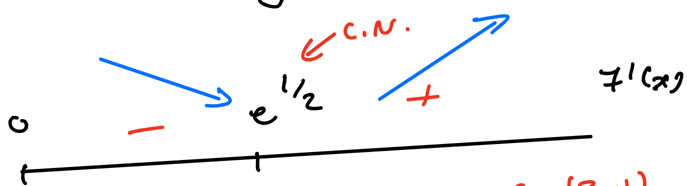
$$\Rightarrow \quad f'(x) = \frac{x\,(2\ln(x) - 1)}{(\ln(x))^2} \quad \leftarrow \text{domain} = (0, \infty)$$

A/ $f'(x) = 0 \Rightarrow x = 0$ or $2\ln(x) - 1 = 0$

$\Rightarrow \ln(x) = \frac{1}{2}$

$\Rightarrow x = e^{1/2}$

B/ $f'$ continuous everywhere on $(0, \infty)$




$$f'(e) = \frac{e\cdot(2-1)}{1^2} > 0$$

$$f'(e^{1/4}) = \frac{e^{\frac{1}{4}}\left(2\cdot\frac{1}{4} - 1\right)}{\left(\frac{1}{4}\right)^2} < 0$$

$$\Rightarrow \quad f(e^{1/2}) = \frac{e}{\left(\frac{1}{2}\right)} = 2e \quad \text{rel. min.}$$

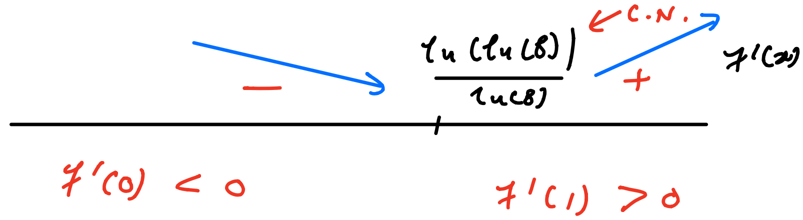
34/ $f(x) = x + 8^{-x} \Rightarrow f'(x) = 1 + \ln(8)\, 8^{-x}\cdot(-1)$

A/ $f'(x) = 0 \Rightarrow 8^{-x} = \frac{1}{\ln(8)} \Rightarrow -x\cdot\ln(8) = \ln\left(\frac{1}{\ln(8)}\right)$

$\Rightarrow -x\ln(8) = -\ln(\ln(8))$

$\Rightarrow x = \frac{\ln(\ln(8))}{\ln(8)} \approx 0.35$

B/ $f'$ continuous everywhere




$$\Rightarrow \quad f\left(\frac{\ln(\ln(8))}{\ln(8)}\right) = \frac{\ln(\ln(8))}{\ln(8)}\, 8^{-\left(\frac{\ln(\ln(8))}{\ln(8)}\right)} \quad \text{rel. min.}$$

48/ $P(x) = \ln(-x^3+3x^2+72x+1)$ on $[0,10]$

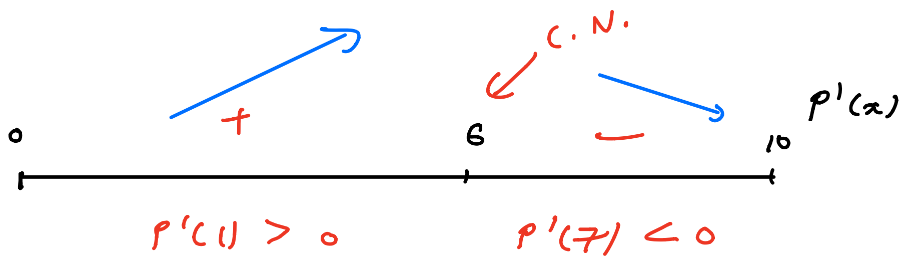
( assume $-x^3+3x^2+72x+1 > 0$ on $[0,10]$ )

$$P'(x) = \frac{-3x^2+6x+72}{-x^3+3x^2+72x+1} = \frac{-3(x^2-2x-24)}{-x^3+3x^2+72x+1}$$

$$= \frac{-3(x-6)(x+4)}{-x^3+3x^2+72x+1}$$

A/ $P'(x) = 0 \Rightarrow x = 6$ or $-4$

B/ $-x^3+3x^2+72x+1 \neq 0$ on $[0,10] \Rightarrow$ No points




$\Rightarrow$ Max profit when $x = 6$

$P(6) = \ln(-6^3+3\,6^2+72\cdot 6+1)$.

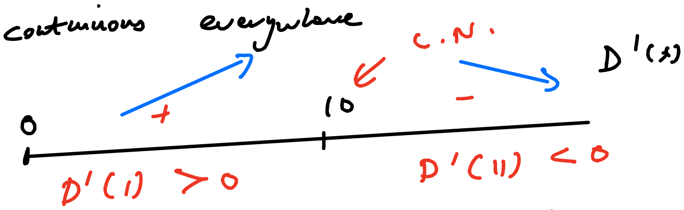
57/ $D(x) = -x^4 + 8x^3 + 80x^2 \quad (x \geq 0)$

$\Rightarrow D'(x) = -4x^3 + 24x^2 + 160x = -4x(x^2 - 6x - 40)$

$= -4x(x-10)(x+4)$

A/ $D'(x) = 0 \Rightarrow x = 0, 10, -4$

B/ $D'$ continuous everywhere




The speaker should aim for a discrepancy at $x = 10$.

58/ $R = \frac{u}{v}$, $u = 20t$, $v = t^2 + 100 \quad (t \geq 0)$

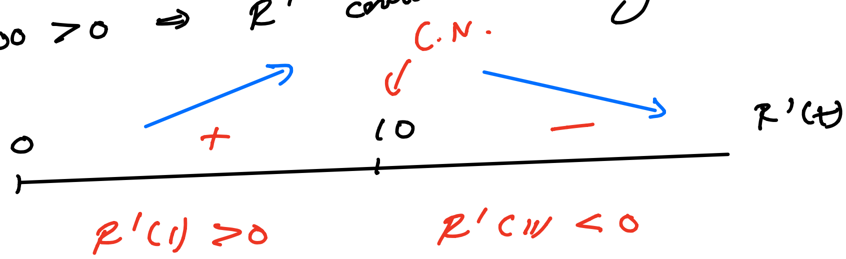
$\Rightarrow \frac{du}{dt} = 20$, $\frac{dv}{dt} = 2t$

$\Rightarrow R'(t) = \frac{\frac{du}{dt}v - u\frac{dv}{dt}}{v^2} = \frac{20(t^2+100) - 20t \cdot (2t)}{(t^2+100)^2}$

$= \frac{-20t^2 + 2000}{(t^2+100)^2}$

A/ $R'(t) = 0 \Rightarrow -20t^2 + 2000 = 0 \Rightarrow t^2 = 100 \Rightarrow t = \pm 10$

B/ $t^2 + 100 > 0 \Rightarrow R'$ continuous everywhere




10 minute films get highest rating.