MATH 16A FINAL EXAM (PRACTICE 3)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ____________________

GSI's name: ____________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. Calculate the following derivatives (you do not need to use limits):

(a)

$$\frac{d}{dx}(2^x - x^2)$$

Solution:

$$\frac{d}{dx}(2^x - x^2) = \ln(2)2^x - 2x$$

(b)

$$\frac{d}{du}(u^{1/3}\ln(u))$$

Solution:

$$\frac{d}{du}(u^{1/3}\ln(u)) = \frac{1}{3}u^{-\frac{2}{3}}\ln(u) + u^{1/3}\cdot\frac{1}{u}$$

(c)

$$\frac{d^2}{dx^2}(\ln(\frac{(x-1)4^x}{x+1}))$$

Solution:

$$\ln\left(\frac{(x-1)4^x}{x+1}\right) = \ln(x-1) + \ln(4)x - \ln(x+1)$$

$$\Rightarrow \frac{d}{dx}\left(\ln\left(\frac{(x-1)4^x}{x+1}\right)\right) = \frac{1}{x-1} + \ln(4) - \frac{1}{x+1}$$

$$\Rightarrow \frac{d^2}{dx^2}\left(\ln\left(\frac{(x-1)4^x}{x+1}\right)\right) = \frac{-1}{(x-1)^2} + \frac{1}{(x+1)^2}$$

2. Calculate the following integrals:

(a)

$$\int \sqrt{2x-1}dx$$

Solution:

$$u = 2x-1 \Rightarrow \frac{du}{dx} = 2 \Rightarrow dx = \frac{du}{2} \Rightarrow$$

$$\int \sqrt{2x-1}\, dx = \int \frac{1}{2}\sqrt{u}\, du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + c = \frac{1}{3}(2x-1)^{3/2} + C$$

(b)

$$\int \frac{2^x}{2^x+1}dx$$

Solution:

$$u = 2^x+1 \Rightarrow \frac{du}{dx} = \ln(2)2^x \Rightarrow dx = \frac{du}{\ln(2)2^x}$$

$$\Rightarrow \int \frac{2^x}{2^x+1}\, dx = \int \frac{1}{\ln(2)}\cdot\frac{1}{u}\, du = \frac{1}{\ln(2)}\ln|2^x+1| + C$$

(c)

$$\int_1^2 \frac{\sqrt[3]{\ln(x)+3}}{x}dx$$

Solution:

$$u = \ln(x)+3 \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du \Rightarrow$$

$$\int \frac{\sqrt[3]{\ln(x)+3}}{x}\, dx = \int u^{1/3}\, du = \frac{3}{4}u^{4/3} + C$$

$$= \frac{3}{4}(\ln(x)+3)^{\frac{4}{3}} + C$$

$$\Rightarrow \int_1^2 \frac{\sqrt[3]{\ln(x)+3}}{x}\, dx = \frac{3}{4}(\ln(x)+3)^{\frac{4}{3}}\Big|_1^2 = \frac{3}{4}(\ln(2)+3)^{\frac{4}{3}} - \frac{3}{4}(3)^{\frac{4}{3}}$$

3. Using the limit definition, calculate the derivative of $f(x) = \frac{1}{\sqrt{x}}$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h \sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{-h}{h \sqrt{x}\sqrt{x+h}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\,(\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x}\,(\sqrt{x}+\sqrt{x})} = \frac{-1}{2x^{3/2}}$$

4. A product is being sold. The supply equation is given by

$$p = q^2 + 2q + 1.$$

The demand equation is given by

$$p = \frac{1000}{q+1}.$$

(a) Calculate the elasticity at the equilibrium.

Solution:

$$p = q^2 + 2q + 1 = (q+1)^2$$

equilibrium quantity

$$(q+1)^2 = \frac{1000}{(q+1)} \Rightarrow (q+1)^3 = 1000 \Rightarrow q+1 = 10 \Rightarrow q = 9$$

equilibrium price

$$\Rightarrow p = 100$$

$$p = \frac{1000}{q+1} \Rightarrow q+1 = \frac{1000}{p} \Rightarrow q = \frac{1000}{p} - 1$$

$$\frac{dq}{dp} = \frac{-1000}{p^2} \Rightarrow E = \frac{-p}{q} \cdot \frac{dq}{dp} = \frac{1000}{pq}$$

$$\Rightarrow \text{Elasticity at equilibrium} = \frac{1000}{9 \times 100} = \frac{10}{9}$$

(b) Calculate the consumer surplus.

Solution:

$$\int_0^9 \frac{1000}{q+1}\,dq = 1000 \ln|q+1| \Big/_0^9 = 1000 \ln 10$$

$$\Rightarrow \text{Consumer Surplus} = 1000 \ln(10) - 900$$

5. Find the equation of the tangent line at $(2, 1)$ of the following curve:

$$3(x^2 + y^2)^2 = 25(x^2 - y^2)$$

Solution:

$$\frac{d}{dx} 3(x^2+y^2)^2 = \frac{d}{dx} 25(x^2-y^2)$$

$$= 6(x^2+y^2) \cdot \left(2x + 2y\frac{dy}{dx}\right) = 50x - 50y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{50x - 12x(x^2+y^2)}{50y + 12y(x^2+y^2)}$$

At $x=2$, $y=1$ $\quad \frac{dy}{dx} = \frac{100 - 120}{50 + 60} = \frac{-2}{11}$

$\Rightarrow$ Tangent line at $(2,1)$: $\quad (y-1) = \frac{-2}{11}(x-2)$

6. Determine the relative extrema of the following function:

$$f(x) = xe^{(x^2-3x)}.$$

Are there any absolute extrema?

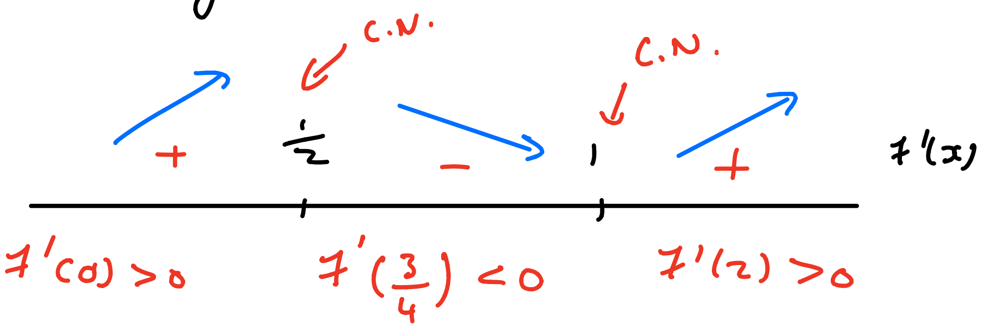
Solution:

$$f'(x) = e^{(x^2-3x)} + x\cdot(2x-3)\,e^{(x^2-3x)}$$

$$= (1 + 2x^2 - 3x)\,e^{(x^2-3x)} = (2x-1)(x-1)\,e^{(x^2-3x)}$$

A/ $f'(x) = 0 \Rightarrow x = 1$ or $1/2$

B/ f' continuous everywhere



$$\Rightarrow f\left(\tfrac{1}{2}\right) = \tfrac{1}{2}e^{\left(\frac{1}{4}-\frac{3}{2}\right)} = \tfrac{1}{2}e^{-\frac{5}{4}} \quad \text{rel. max}$$

$$f(1) = e^{-2} \quad \text{rel. min}$$

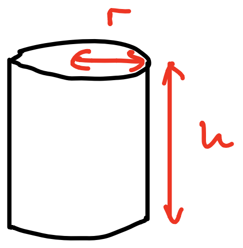
$$f(0) = 0 < e^{-2} = f(1) \Rightarrow \text{No absolute min}$$

$$f(10) = 10e^{70} > \tfrac{1}{2}e^{-\frac{5}{2}} \Rightarrow \text{No absolute max}$$

7. A drink will be packaged in cylindrical cans with volume 40in^3. The top and bottom of the can cost 4 cents per square inch. The sides cost 3 cents per square inch. Determine the dimensions of the can which minimize costs.

Solution:

Objective : Minimize costs



cost of top and bottom; cost of sides

Objective : $\underbrace{4\pi r^2 + 4\pi r^2}_{\text{cost of top and bottom}} + \underbrace{3 \cdot 2\pi r h}_{\text{cost of sides}}$

Constraint : $\pi r^2 h = 40$

$\Rightarrow \quad h = \dfrac{40}{\pi r^2}$

$$\Rightarrow \quad 8\pi r^2 + 6\pi r h = 8\pi r^2 + 6\pi r \cdot \frac{40}{\pi r^2}$$

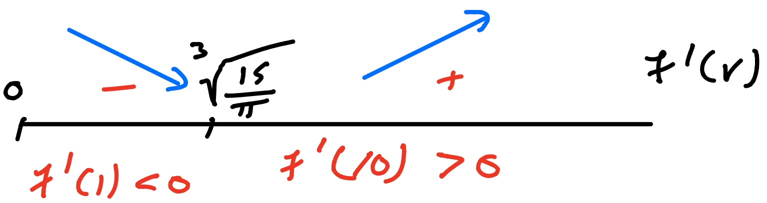
$$= 8\pi r^2 + \frac{240}{r} = f(r)$$

Domain : $(0, \infty)$

$f'(r) = 16\pi r - \dfrac{240}{r^2}$

A/ $f'(r) = 0 \Rightarrow 16\pi r = \dfrac{240}{r^2} \Rightarrow r^3 = \dfrac{\quad}{\pi} \Rightarrow r = \sqrt[3]{\dfrac{15}{\pi}}$

B/ f' continuous on $(0, \infty)$



$\Rightarrow$ Absolute min cost is when $r = \sqrt[3]{\dfrac{15}{\pi}}$, $h = \dfrac{40}{\pi\left(\frac{15}{\pi}\right)^{\frac{2}{3}}}$

8. Sketch the following curve. If they exist, be sure to indicate relative extrema and inflection points. Show your working on this page and draw the graph on the next page.

$$y = \frac{3x}{x-2} = f(x)$$

Solution:

Domain : $x \neq 2$

$f(0) = 0 \Rightarrow (0,0) =$ x and y intercept.

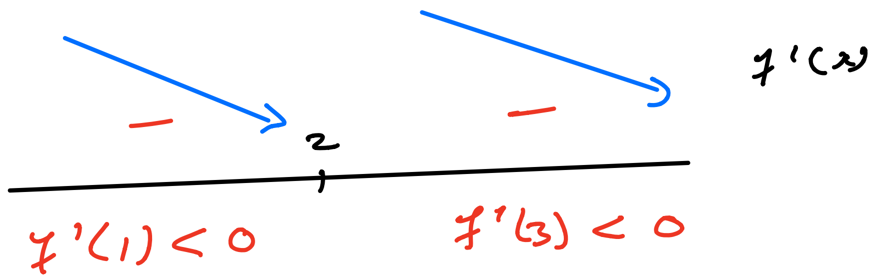
$\lim_{x \to \pm\infty} \frac{3x}{x-2} = 3 \Rightarrow y = 3$ is horizontal asymptote

$x = 2$ vertical asymptote

$$f'(x) = \frac{3(x-2) - 3x}{(x-2)^2} = \frac{-6}{(x-2)^2}$$

A/ $f'(x) = 0$ No solution

B/ f' undefined when $x = 2$



$$f''(x) = \frac{12}{(x-2)^3}$$

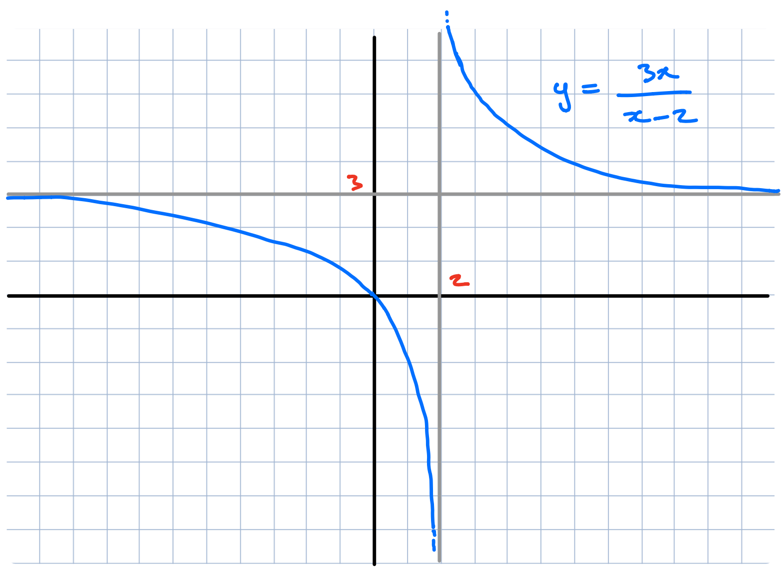
A/ $f'(x) = 0$ No solutions

B/ f' undefined at $x = 2$

$f'(2)$ undefined (not inflection)

$-$ $\quad$ 2 $\quad$ $+$

$f''(1) < 0$ $\quad$ $f''(3) > 0$



9. Let $f(x) = x^5 - 2\ln((x+20)^3)$ and $g(x) = x^3 - 6\ln(x+20)$. Calculate the area of the region bounded by $y = f(x)$ and $y = g(x)$.

Solution:

$$f(x) = g(x) \Rightarrow x^5 - 2\ln((x+20)^3) = x^3 - 6\ln(x+20)$$

$$\Rightarrow x^5 = x^3 \qquad \left(2\ln((x+20)^3) = 6\ln(x+20)\right)$$

$$\Rightarrow x^3(x^2-1) = 0 \Rightarrow x = 0, 1, -1$$

$$\int_{-1}^{0} x^5 - x^3 \, dx = \frac{1}{6}x^6 - \frac{1}{4}x^4 \Big|_{-1}^{0}$$

$$= 0 - \left(\frac{1}{6} - \frac{1}{4}\right) = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

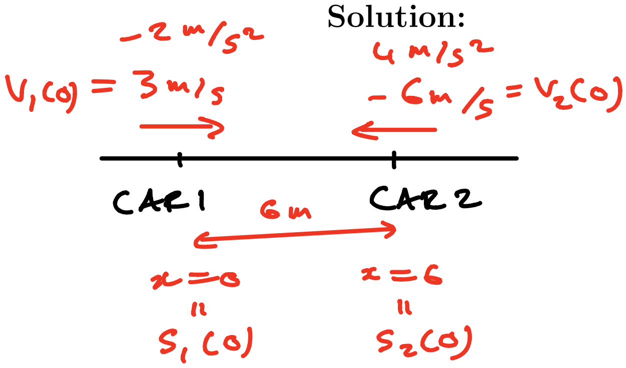
$$\int_{0}^{1} x^5 - x^3 \, dx = \frac{1}{6}x^6 - \frac{1}{4}x^4 \Big|_{0}^{1} = \frac{1}{6} - \frac{1}{4} = \frac{-1}{12}$$

$\Rightarrow$ Total area enclosed $= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$.

10. Two cars are travelling directly towards each other on a straight road. The first car is travelling at 3 metres per second. The second car is travelling at 6 metres per second. When they are 6 metres apart they simultaneously apply the brakes. The first car decelerates at a constant rate of 2 metres per second per second. The second car decelerates at a constant rate of 4 metres per second per second.

(a) (15 points) How long after applying the brakes will the cars collide? Carefully justify your answer. Hint: If t is the time in seconds after they both apply the brakes, first calculate $s_1(t)$ and $s_2(t)$, position functions for the first and second car respectively.

Solution:



$$a_1(t) = -2 \qquad v_1(t) = -2t + 3$$
$$\Rightarrow$$
$$a_2(t) = 4 \qquad v_2(t) = 4t - 6$$

$$\Rightarrow \quad s_1(t) = -t^2 + 3t \qquad (s_1(0) = 0)$$
$$s_2(t) = 2t^2 - 6t + 6 \qquad (s_2(0) = 6)$$

$$s_1(t) = s_2(t) \Rightarrow -t^2 + 3t = 2t^2 - 6t + 6$$
$$\Rightarrow 3t^2 - 9t + 6 = 0 \Rightarrow 3(t-1)(t-2) = 0$$
$$\Rightarrow t = 1 \text{ or } 2$$

$1 < 2 \Rightarrow$ Cars collide when $t = 1$

(b) (5 points) Determine the velocity of each car when they collide.

Solution:

$$v_1(1) = 1 \text{ m/s}$$
$$v_2(1) = -2 \text{ m/s}$$

END OF EXAM