MATH 16A FINAL EXAM (PRACTICE 2)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. Calculate the following derivatives (you do not need to use limits):

(a)

$$\frac{d}{dx}(x\sqrt{x}+3)$$

Solution:

$$\frac{d}{dx}(x^{3/2}+3) = \frac{3}{2}x^{1/2}$$

(b)

$$\frac{d}{dt}\left(\frac{t^2+1}{t^3-2}\right)$$

Solution:

$$\frac{d}{dt}\left(\frac{t^2+1}{t^3-2}\right) = \frac{2t(t^3-2) - (t^2+1)(3t^2)}{(t^3-2)^2}$$

(c)

$$\frac{d^2}{dx^2}(3^{\sqrt{x}})$$

Solution:

$$\frac{d}{dx}(3^{\sqrt{x}}) = \ln(3)\,\frac{1}{2}x^{-\frac{1}{2}}\,3^{\sqrt{x}}$$

$$\Rightarrow \frac{d^2}{dx^2}(3^{\sqrt{x}}) = \ln(3)\cdot\frac{-1}{4}x^{-\frac{3}{2}}\cdot 3^{\sqrt{x}} + \left(\ln(3)\,\frac{1}{2}x^{-\frac{1}{2}}\right)^2 3^{\sqrt{x}}$$

2. Calculate the following integrals:

(a)

$$\int (x^3 - x)dx$$

Solution:

$$\int x^3 - x\,dx = \frac{1}{4}x^4 - \frac{1}{2}x^2 + C$$

(b)

$$\int \frac{\sqrt{x}-1}{\sqrt{x}}dx$$

Solution:

$$\int \frac{\sqrt{x}-1}{\sqrt{x}}\,dx = \int 1 - x^{-\frac{1}{2}}\,dx = x - \frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1} + C$$

$$= x - 2\sqrt{x} + C$$

(c)

$$\int_1^2 \frac{2^{(1/u)}}{u^2}du$$

Solution:

$$v = 1/u \Rightarrow \frac{dv}{du} = \frac{-1}{u^2} \Rightarrow du = -u^2\,dv \Rightarrow$$

$$\int \frac{2^{(1/u)}}{u^2}\,du = \int -2^v\,dv = \frac{-1}{\ln(2)}2^v + C = \frac{-1}{\ln(2)}2^{(1/u)} + C$$

$$\Rightarrow \int_1^2 \frac{2^{(1/u)}}{u^2}\,du = \frac{-1}{\ln(2)}2^{1/u}\Big|_1^2 = \left(\frac{-1}{\ln(2)}2^{1/2}\right) - \left(\frac{-2}{\ln(2)}\right)$$

3. Using the limit definition, calculate the derivative of $f(x) = \frac{2}{x^2} + x$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{\left(\frac{2}{(x+h)^2} + (x+h)\right) - \left(\frac{2}{x^2} + x\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{(x+h)^2} - \frac{2}{x^2} + h}{h}$$

$$= \lim_{h \to 0} \frac{2x^2 - 2(x+h)^2}{h(x+h)^2 x^2} + 1$$

$$= \lim_{h \to 0} \frac{-4xh - 2h^2}{h(x+h)^2 x^2} + 1$$

$$= \lim_{h \to 0} \frac{-4x - 2h}{(x+h)^2 x^2} + 1$$

$$= \frac{-4x}{x^4} + 1 = -4x^{-3} + 1$$

4. A product is being sold. The demand equation is $q^2 + qp + 4p^2 = 10$, where p is the price per unit and q is the number of units sold.

   (a) Calculate the elasticity. Your answer should involve both p and q.
   Solution:

$$\frac{d}{dp}\left(q^2 + qp + 4p^2\right) = \frac{d}{dp}(10)$$

$$\Rightarrow 2q\frac{dq}{dp} + \frac{dq}{dp}\cdot p + q + 8p = 0$$

$$\Rightarrow \frac{dq}{dp} = \frac{-8p - q}{2q + p}$$

$$\Rightarrow \text{Elasticity} = \frac{-p}{q}\cdot\frac{-8p - q}{2q + p} = \frac{8p^2 + pq}{2q^2 + pq}$$

   (b) If $q = 3$ should they increase or decrease the price to raise revenue?
   Solution:

$$q = 3 \Rightarrow 3^2 + 3p + 4p^2 = 10 \Rightarrow 4p^2 + 3p - 1 = 0$$

$$\Rightarrow (4p-1)(p+1) = 0 \Rightarrow p = \frac{1}{4} \quad (p > 0)$$

$$\frac{8\cdot\left(\frac{1}{4}\right)^2 + \frac{3}{4}}{18 + \frac{3}{4}} < 1 \Rightarrow \text{Inelastic so they should raise the price.}$$

5. Determine the concavity of the following function:

$$f(x) = x^2 + 8\ln|x+1|.$$

Are there any inflection points? If so, find them.

Solution:

$$f'(x) = 2x + \frac{8}{x+1}$$

$$f''(x) = 2 - \frac{8}{(x+1)^2}$$

A/ $f''(x) = 0 \Rightarrow (x+1)^2 = 4 \Rightarrow x+1 = \pm 2 \Rightarrow x = -3$ or 1

B/ f'' undefined when $x = -1$

$\Rightarrow$ $y = f(x)$ concave up on $(-\infty, -3)$ and $(1, \infty)$

$y = f(x)$ concave down on $(-3, -1)$ and $(-1, 1)$

There are inflection points at $x = -3$ and $x = 1$

6. Sketch the following curve. If they exist, be sure to indicate relative extrema and inflection points. Show your working on this page and draw the graph on the next page.

$$y = \frac{1}{x^2 + 4x + 3}$$

Solution:

$$f(x) = \frac{1}{x^2+4x+3} = \frac{1}{(x+3)(x+1)}$$

Domain : $x \neq -3, -1$

$f(0) = \frac{1}{3} \Rightarrow (0, \frac{1}{3}) =$ y-intercept

$f(x) = 0$ no solutions $\Rightarrow$ no x-intercept

$\lim_{x \to \pm\infty} f(x) = 0 \Rightarrow y = 0$ horizontal asymptote

$x = -1$, $x = -3$ vertical asymptotes

$$f'(x) = \frac{-(2x+4)}{(x^2+4x+3)^2}$$

A/ $f'(x) = 0 \Rightarrow x = -2$

B/ f' undefined $\Rightarrow x = -1, -3$

C.N.

$f'(x)$: $+$ $\;-3\;$ $+$ $\;-2\;$ $-$ $\;-1\;$ $-$

$f'(-4) > 0$ $\quad f'(-\frac{5}{2}) > 0$ $\quad f'(-\frac{3}{2}) < 0$ $\quad f'(0) < 0$

$f(-2) = -1$

$$f''(x) = \frac{(-2)(x^2+4x+3)^2 - (-(2x+4))\cdot(2x+4)\cdot 2\cdot(x^2+4x+3)}{(x^2+4x+3)^4}$$

$$= \frac{-2(x^2+4x+3) + 2(2x+4)^2}{(x^2+4x+3)^3}$$

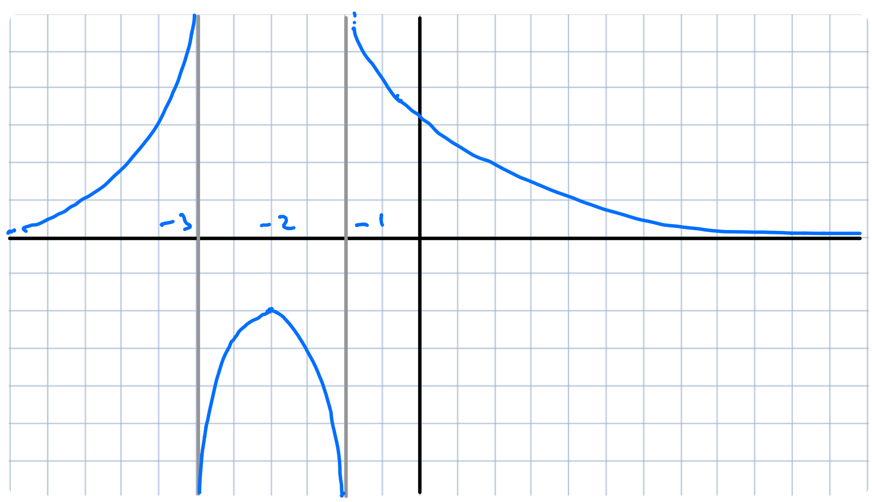
$$= \frac{6x^2+24x+26}{(x+1)^3(x+3)^3}$$

No solution ($b^2-4ac<0$)

A/ $f''(x)=0 \Rightarrow 6x^2+8x+26 = 0$

B/ f'' undefined $\Rightarrow x = -1, -3$

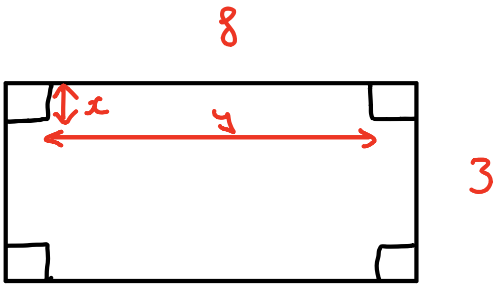
$+$	-3	$-$	-1	$+$	$f''(x)$
$f''(-4)>0$		$f''(-2)<0$		$f''(0)>0$	



7. An open box will be made by cutting a square from each corner of a 3 by 8 foot piece of cardboard and then folding up the sides. What size squares should be cut from each corner to maximize the volume?

Solution:

Objective : Maximize Volume



Objective : Volume $= y \cdot x \cdot (3-2x)$

Constraint : $2x + y = 8$

$\Rightarrow \quad y = 8 - 2x$

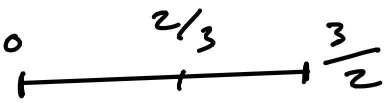
$\Rightarrow$ Volume $= (8-2x)\, x\, (3-2x) = +4x^3 - 22x^2 + 24x$
$= f(x)$

Domain : $[0, \frac{3}{2}]$

$f'(x) = 12x^2 - 44x + 24 = 3x^2 - 11x + 6$
$= (3x-2)(x-3)$

A) $f'(x) = 0 \Rightarrow x = 2/3$ or 3

B) f' continuous everywhere



$f(0) = 0$
$f(3/2) = 0$
$f(2/3) > 0$

$\Rightarrow$ Volume is maximized when $x = 2/3$ (8 inches)

8. A company incurs debt at a rate of

$$(2t+3)\sqrt{t+1}$$

dollars per year, where t is the amount of time (in years) since the company started. How much will the company's debts have grown between $t = 3$ and $t = 8$?

Solution:

$$D'(t) = (2t+3)\sqrt{t+1}$$

$$u = t+1 \Rightarrow \frac{du}{dt} = 1 \Rightarrow dt = du$$

$$(\Rightarrow t = u-1)$$

$$\Rightarrow \int (2t+3)\sqrt{t+1}\, dt = \int (2t+3)\sqrt{u}\, du$$

$$= \int (2(u-1)+3)\sqrt{u}\, du = \int 2u^{3/2} + u^{1/2}\, du$$

$$= 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C = \frac{4}{5}(t+1)^{5/2} + \frac{2}{3}(t+1)^{3/2} + C$$

$$\Rightarrow \int_3^8 (2t+3)\sqrt{t+1}\, dt = \frac{4}{5}(t+1)^{5/2} + \frac{2}{3}(t+1)^{3/2}\Big|_3^8$$

$$= \left(\frac{4}{5}\cdot 3^5 + \frac{2}{3}3^3\right) - \left(\frac{4}{5}2^5 + \frac{2}{3}2^3\right)$$

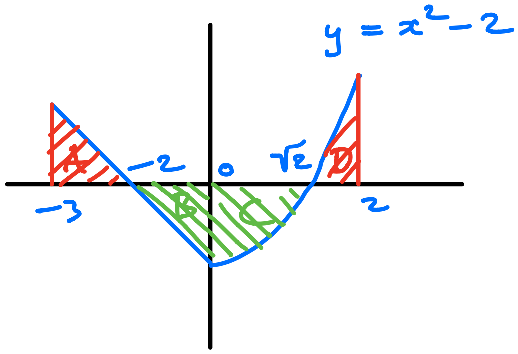
$$= D(8) - D(3)$$

9. Determine the area of the region enclosed by the x-axis and the curve

$$y = \begin{cases} -2 - x & \text{if } x < 0 \\ x^2 - 2 & \text{if } x \geq 0 \end{cases}$$

between -3 and 2.

Solution:



$$\text{Area}(A) = \int_{-3}^{-2} -2 - x \, dx = -2x - \frac{1}{2}x^2 \Big|_{-3}^{-2}$$

$$= (4-2) - \left(6 - \frac{9}{2}\right)$$

$$= \frac{1}{2}$$

$$\text{Area}(B) = -\int_{-2}^{0} -2 - x \, dx = 2x + \frac{1}{2}x^2 \Big|_{-2}^{0}$$

$$= 0 - (-4+2) = 2$$

$$\text{Area}(C) = -\int_{0}^{\sqrt{2}} x^2 - 2 \, dx = 2x - \frac{1}{3}x^3 \Big|_{0}^{\sqrt{2}}$$

$$= 2\sqrt{2} - \frac{2}{3}\sqrt{2} = \frac{4}{3}\sqrt{2}$$

$$\text{Area}(D) = \int_{\sqrt{2}}^{2} x^2 - 2 \, dx = \frac{1}{3}x^3 - 2x \Big|_{\sqrt{2}}^{2} = \left(\frac{8}{3} - 4\right) - \left(\frac{-4}{3}\sqrt{2}\right)$$

$$\Rightarrow \text{Total Area} = \frac{1}{2} + 2 + \frac{4}{3}\sqrt{2} + \frac{4}{3}\sqrt{2} - \frac{4}{3}.$$

10. Let $f(x) = x^4 + x^3 + x^2 - 2x + 1$ and $g(x) = x^4 + x^2 - x + 1$. Calculate the total area of the region bounded by $y = f(x)$ and $y = g(x)$.

Solution:

$$f(x) = g(x) \Rightarrow x^4 + x^3 + x^2 - 2x + 1 = x^4 + x^2 - x + 1$$

$$\Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) \Rightarrow x = 0, 1, -1$$

$$\int_0^1 x^3 - x\,dx = \frac{1}{4}x^4 - \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{4} - \frac{1}{2} = \frac{-1}{4}$$

$$\int_{-1}^0 x^3 - x\,dx = \frac{1}{4}x^4 - \frac{1}{2}x^2 \Big|_{-1}^0 = 0 - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= \frac{1}{4}$$

$\Rightarrow$ Total area enclosed is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

END OF EXAM