MATH 16A FINAL EXAM (PRACTICE 1)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. Calculate the following derivatives (you do not need to use limits):

   (a)

$$\frac{d}{dx}(4x^3+7)$$

Solution:

$$\frac{d}{dx}(4x^3+7) = 12x^2$$

   (b)

$$\frac{d}{ds}(\log_2(2s^3+2^s))$$

Solution:

$$\frac{d}{ds}(\log_2(2s^3+2^s)) = \frac{6s^2 + \ln(2)2^s}{\ln(2)\cdot(2s^3+2^s)}$$

   (c)

$$\frac{d^2}{dx^2}(e^{(1/2x)})$$

Solution:

$$\frac{d}{dx}(e^{(1/2x)}) = \frac{1}{2}\cdot\frac{-1}{x^2}\,e^{(1/2x)}$$

$$\Rightarrow \frac{d^2}{dx^2}(e^{(1/2x)}) = \frac{1}{x^3}e^{(1/2x)} + \frac{1}{4x^4}e^{(1/2x)}$$

2. Calculate the following integrals:

(a)

$$\int (x^2 + 4\sqrt{x})dx$$

Solution:

$$\int x^2 + 4\sqrt{x}\, dx = \frac{1}{3}x^3 + 4 \cdot \frac{2}{3} x^{3/2} + C$$

(b)

$$\int \frac{e^{3x} + 3^x}{e^x} dx$$

Solution:

$$\int \frac{e^{3x} + 3^x}{e^x} dx = \int e^{2x} + \left(\frac{3}{e}\right)^x dx = \frac{1}{2}e^{2x} + \frac{1}{\ln\left(\frac{3}{e}\right)}\left(\frac{3}{e}\right)^x + C$$

(c)

$$\int_1^4 \frac{2^{(\sqrt{x})}}{5\sqrt{x}} dx$$

Solution:

$$u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow dx = 2\sqrt{x}\, du \Rightarrow$$

$$\int \frac{2^{(\sqrt{x})}}{5\sqrt{x}} dx = \int \frac{2}{5} 2^u du = \frac{2}{5\ln(2)} 2^u + C = \frac{2}{5\ln(2)} 2^{\sqrt{x}} + C$$

$$\Rightarrow \int_1^4 \frac{2^{(\sqrt{x})}}{5\sqrt{x}} dx = \frac{2}{5\ln(2)} 2^{\sqrt{x}} \Big|_1^4 = \frac{2}{5\ln(2)} (2^2 - 2^1)$$

3. A product is to be made and sold. Both the cost and the revenue functions are linear. The marginal cost is 3 and the fixed costs are 5. If two items are made and sold there is a loss of 3. If six items are made and sold there is a profit of 1.

   (a) Determine the revenue and cost functions.

   Solution:

marginal cost   fixed cost

$C(x) = 3x + 5$

$R(x) = mx + b$

$P(2) = R(2) - C(2) = 2m + b - 11 = -3 \Rightarrow b = 8 - 2m$

$P(6) = R(6) - C(6) = 6m + b - 23 = 1 \Rightarrow b = 24 - 6m$

$\Rightarrow 8 - 2m = 24 - 6m \Rightarrow 4m = 16 \Rightarrow m = 4 \Rightarrow b = 0$

$\Rightarrow R(x) = 4x$

   (b) Determine the break-even quantity.

   Solution:

$C(x) = R(x) \Rightarrow 3x + 5 = 4x \Rightarrow x = 5$ is break-even quantity.

4. Using the limit definition, calculate the derivative of $f(x) = \sqrt{x^2+1}$.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \cdot \frac{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)} = \lim_{h\to 0} \frac{2xh + h^2}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h\to 0} \frac{2x + h}{\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)} = \frac{2x}{2\sqrt{x^2+1}}$$

5. Find the equation of the tangent line at $x = 3$ of the following curve:

$$2y^3(x-3) + x\sqrt{y} = 3.$$

Solution:

$$\frac{d}{dx}\left(2y^3(x-3) + x\sqrt{y}\right) = \frac{d}{dx}(3)$$

$$\Rightarrow 6y^2\frac{dy}{dx}(x-3) + 2y^3 + \sqrt{y} + x\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{y} - 2y^3}{6y^2(x-3) + x\frac{1}{2}y^{-\frac{1}{2}}}$$

$$x = 3 \Rightarrow 3\sqrt{y} = 3 \Rightarrow y = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1-2}{3 \cdot \frac{1}{2}} = \frac{-3}{(3/2)} = -2$$

$\Rightarrow$ Equation of Target at $(3,1)$: $y - 1 = -2(x-3)$

6. You open a savings account where interest is compounded continuously. The balance in the account doubles every 10 years.

   (a) Determine the annual interest rate (as a percentage). You do not need to simplify your answer.

   Solution:

$A(t)$ = Balance of account at time t (in years)

$A(t) = A_0 e^{rt}$ (A_0 = initial investment, r = interest rate)

$A(10) = 2A(0) = 2A_0 \Rightarrow A_0 e^{10r} = 2A_0$

$\Rightarrow e^{10r} = 2 \Rightarrow r = \frac{\ln(2)}{10}$

$\Rightarrow$ Annual percentage interest rate is $\frac{\ln(2)}{10} \times 100 = 10\ln(2)\,\%$

   (b) After three years the account balance is \$1000. Determine the initial investment. You do not need to simplify your answer.

   Solution:

$A(3) = 1000 \Rightarrow A_0 e^{3\cdot\frac{\ln(2)}{10}} = 1000$

$\Rightarrow A_0 = \$\frac{1000}{e^{\frac{3\ln(2)}{10}}}$

7. Consider the function $f(x) = 4x + \frac{1}{\sqrt{x}} + 1$, where $x > 0$.

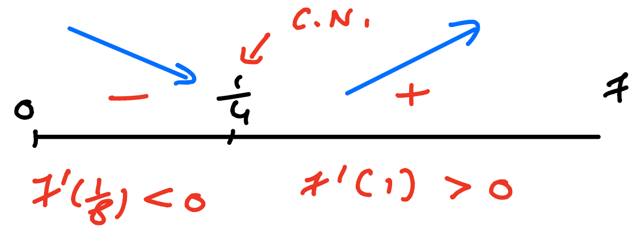
(a) Determine all relative maxima and minima of this function. Does the function have absolute maxima or minima? Carefully justify your answer.

Solution:

$$f'(x) = 4 - \frac{1}{2x^{3/2}}$$

A/ $f'(x) = 0 \Rightarrow x^{3/2} = \frac{1}{8} \Rightarrow x = \frac{1}{4}$

B/ f' continuous on $(0, \infty)$



$\Rightarrow f(\frac{1}{4}) = 1 + 2 + 1 = 4$ relative min.

It is also an absolute min.

There are no rel/abs maxima.

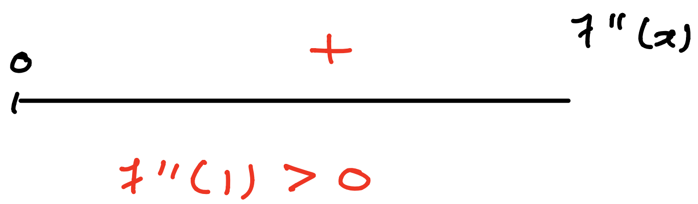
(b) Determine where the graph $y = f(x)$ is concave up. Does the graph have any inflection points? Carefully justify your answer.

Solution:

$$f''(x) = \frac{3}{4x^{5/2}}$$

A/ $f''(x) = 0$ No solution

B/ f'' continuous on $(0, \infty)$



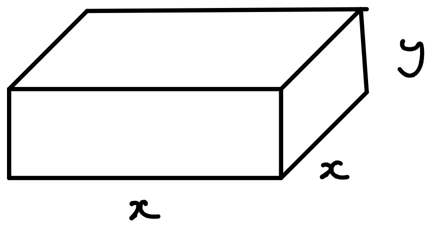
$\Rightarrow y = f(x)$ concave up on $(0, \infty)$

There are no inflection points.

8. Determine the minimum possible surface area of a closed box with a square base and volume 1000cm^3

Solution:

Objective : Minimize Surface Area



Objective : $2x^2 + 4xy$

Constraint : $x^2 y = 1000$

$\Rightarrow \quad y = \dfrac{1000}{x^2}$

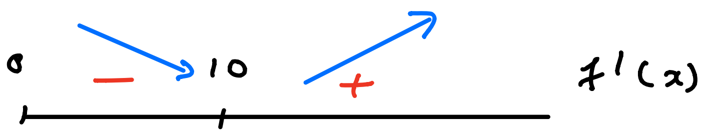
$$\Rightarrow \quad 2x^2 + 4x \cdot \frac{1000}{x^2} = 2x^2 + \frac{4000}{x} = f(x)$$

Domain : $(0, \infty)$

$$f'(x) = 4x - \frac{4000}{x^2}$$

A/ $f'(x) = 0 \Rightarrow x^3 = 1000 \Rightarrow x = 10$

B/ f' continuous on $(0, \infty)$



$f'(1) < 0 \quad f'(11) > 0$

$\Rightarrow \quad f(10) = 600$ is minimum surface area.

9. Two rockets are fired vertically into the air from the ground. The second rocket is launched four seconds after the first. The velocity of the first rocket is $v_1(t) = 6 - t$ metres per second and the velocity of the second is $v_2(t) = 10 - t$ metres per second, where t is the time in seconds after the first launch.

   (a) (15 points) How long after the launch will both rockets be at the same height? What will this height be?

   Solution:

$s_1(t) = 6t - \frac{1}{2}t^2 + C$ and $s_1(0) = 0 \Rightarrow C = 0$

$\Rightarrow s_1(t) = 6t - \frac{1}{2}t^2$

$s_2(t) = 10t - \frac{1}{2}t^2 + C$ and $s_2(4) = 0 \Rightarrow 40 - 8 + C = 0$

$\Rightarrow C = -32 \Rightarrow s_2(t) = 10t - \frac{1}{2}t^2 - 32$

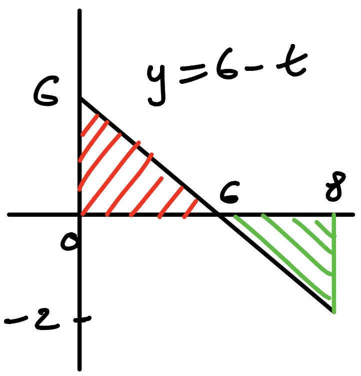
$s_1(t) = s_2(t) \Rightarrow 6t - \frac{1}{2}t^2 = 10t - \frac{1}{2}t^2 - 32$

$\Rightarrow 4t = 32 \Rightarrow t = 8.$ $\leftarrow$ time after 1st lauch they'll be same height

$s_1(8) = s_2(8) = 16$ $\leftarrow$ height at $t=8$

   (b) (10 points) Determine the total distance traveled by the first rocket at this time.

   Solution:



Total Distance travelled

$= \text{Area}(\quad) + \text{Area}(\quad)$

$= \frac{6 \times 6}{2} + \frac{2 \times 2}{2} = 18 + 2 = 20 \text{ m.}$

10. Let $f(x) = 2e^{3x}$ and $g(x) = e^{3x} + e^6$. Calculate the area of the region bounded by $y = f(x)$ and $y = g(x)$ between 0 and 3.

Solution:

$$f(x) = g(x) \Rightarrow 2e^{3x} = e^{3x} + e^6 \Rightarrow e^{3x} = e^6$$

$$\Rightarrow 3x = 6 \Rightarrow x = 2$$

$$\int_0^2 2e^{3x} - (e^{3x} + e^6)\, dx = \int_0^2 e^{3x} - e^6\, dx = \frac{1}{3}e^{3x} - e^6 x \Big|_0^2$$

$$= \left(\frac{1}{3}e^6 - 2e^6\right) - \frac{1}{3} = \frac{-5}{3}e^6 - \frac{1}{3} < 0$$

$$\int_2^3 e^{3x} - e^6\, dx = \frac{1}{3}e^{3x} - e^6 x \Big|_2^3 = \left(\frac{1}{3}e^9 - 3e^6\right) - \left(\frac{1}{3}e^6 - 2e^6\right)$$

$$= \frac{1}{3}e^9 - 3e^6 + \frac{5}{3}e^6$$

$$= \frac{1}{3}e^9 - \frac{4}{3}e^6 > 0$$

$$\Rightarrow \text{Area enclosed} = \frac{1}{3}e^9 - \frac{4}{3}e^6 + \frac{5}{3}e^6 + \frac{1}{3}$$

$$= \frac{1}{3}(e^9 + e^6 + 1)$$

END OF EXAM