Math 113 Homework 7

To be submitted in class on Thursday August 10

Important Information:

- This homework will be graded for completeness and correctness. Two questions will be selected at random for close scrutiny. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make.
- Problems will be of varying difficulty, and do not appear in any order of difficulty. Expect to spend between 5 and 10 hours for each homework set.
- Good Luck!

Question 1

Let $f(x), g(x) \in \mathbb{Q}[X]$ and suppose that $f(x)$ is irreducible. Also suppose that $f(x)$ and $g(x)$ share a common zero in $\mathbb{C}$. Prove that $f(x)$ divides $g(x)$.

Question 2 (Hard)

Let R be a UFD. Show that any irreducible in $R[X]$ is either an irreducible in R or is a primitive polynomial in $R[X]$. Use Gauss' Lemma to show that a primitive polynomial is irreducible in $R[X]$ if and only if it is irreducible in $F[X]$, where $F = Frac(R)$. Deduce that $R[X]$ is a UFD.

Question 3 (Hard)

Let p be a prime number. Prove that $1 + x + \cdots + x^{p-1}$ is irreducible in $\mathbb{Q}[X]$. Is there a way to apply Eisenstein's criterion?

Question 4

Let E/F be a finite extension of infinite fields. Prove that the index of $(F, +)$ in $(E, +)$ is finite if and only if $E = F$.

Question 5

Prove that if $E/\mathbb{C}$ is a finite field extension then $E = \mathbb{C}$. You may assume the Fundamental Theorem of Algebra.

Question 6

Let p be a prime number. Let $\zeta_p = e^{2\pi i/p} \in \mathbb{C}$. Using question 3 prove that $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p - 1$. For $n \in \mathbb{N}$ let $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$. Prove that $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is finite. Let $\mathbb{Q}(\zeta_\infty) := \cup_{n \in \mathbb{N}} \mathbb{Q}(\zeta_n) \subset \mathbb{C}$. Prove that $\mathbb{Q}(\zeta_\infty) \subset \mathbb{C}$ is a subfield. Using this prove that $\mathbb{Q}(\zeta_\infty)$ is algebraic but not finite over $\mathbb{Q}$.