Math 113 — Homework 6

To be submitted in class on Thursday August 3

Important Information:

- This homework will be graded for completeness and correctness. Two questions will be selected at random for close scrutiny. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make.
- Problems will be of varying difficulty, and do not appear in any order of difficulty. Expect to spend between 5 and 10 hours for each homework set.
- Good Luck!

Question 1

Let R be an integral domain of characteristic p, a prime number. Show that the map of sets

$$\begin{array}{rcl} \phi : R & \longrightarrow & R \\ a & \longrightarrow & a^p \end{array}$$

is an endomorphism. Show that if R is finite than ϕ is an automorphism.

Question 2

Let R be a UFD in which the units, together with 0, form a subring $F \subset R$. Show that F is in fact a subfield of R and if $F \neq R$, then R contains infinitely many non-associated primes. (Hint: mimic the usual argument of $\mathbb{Z}$).

Question 3

Consider the subring $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} | a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Determine the units in $\mathbb{Z}[\sqrt{-3}]$.

By considering the factorisations $4 = 2 \cdot 2 = (1+\sqrt{-3})(1-\sqrt{-3})$ prove that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD. If you claim an element irreducible be sure to prove it.

Question 4 (Hard)

Prove that the function sending $a + bi$ to $a^2 + b^2$ makes $\mathbb{Z}[i]$ Euclidean.

Question 5

Find all the zeros of the polynomial $x^2 - 1$ inthe ring $\mathbb{Z}/16\mathbb{Z}$.

Question 6

Determine all irreducible polynomials of degrees 2 and 3 in $\mathbb{F}_2[X]$.

Question 7

Give examples of the following:

1. A ring R where $R^* = R$.
2. A ring R where 1_R has infinite additive order and R contains zero-divisors.
3. An ideal $I \subset \mathbb{C}[X]$ where there exists $f(X)$ such that $f(X)^5 \in I$ but $f(x) \notin I$.

Question 8

Let F be a field and $I \subset F[X]$ be an ideal. Prove that if there exist $f(X), g(X) \in I$ such that $HCF(f(X), g(X)) = 1_{F[X]}$, then $I = F[X]$.

Question 9

For each of the following ideals, say whether they are prime, maximal (hence also prime), or neither

1. $(x^4 + 2x^2 + 1) \subset \mathbb{C}[x]$.
2. $(4, 2x - 1) \subset \mathbb{Z}[x]$. Recall from homework 5 that this is the ideal generated by $\{4, 2x - 1\}$.

Question 10

Prove that if R is an integral domain and (a) is a nonzero prime ideal, then a is an irreducible element.

Prove that if R is a PID and a is irreducible, then (a) is maximal.