Math 113 — Homework 5

To be submitted in class on Thursday July 27

Important Information:

- This homework will be graded for completeness and correctness. Two questions will be selected at random for close scrutiny. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make.
- Problems will be of varying difficulty, and do not appear in any order of difficulty. Expect to spend between 5 and 10 hours for each homework set.
- Good Luck!

Question 1

Let R be a non-trivial commutative ring. Prove that if $I \subset R$ is an ideal then

$$I = R \iff 1_R \in I$$

Prove that R is a field if and only if its only ideals are $\{0_R\}$ and R.

Question 2

Let $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$. Prove that $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{C}$.

Question 3

Give examples of (commutative) subrings R of $\mathbb{C}$ satisfying the following containments and non-containments. If it's not possible, explain why not.

1. $\mathbb{Q} \subset R$, $R \subset \mathbb{R}$ ($\mathbb{Q} \neq R \neq \mathbb{R}$).
2. $\mathbb{Q} \subset R$, $\mathbb{R} \not\subset R$ ($R \neq \mathbb{Q}$).
3. $\mathbb{Z} \subset R$, $\mathbb{Q} \not\subset R$ ($R \neq \mathbb{Z}$).
4. $\mathbb{R} \subset R$, $R \subset \mathbb{C}$ ($\mathbb{R} \neq R \neq \mathbb{C}$).

Question 4

Let R be a ring. We say that $r \in R$ is an idempotent if $r^2 = r$.

Prove that if R is a ring in which every element is an idempotent, then R is in fact commutative, and satisfies $r + r = 0_R$ for every $r \in R$.

Question 5

Decide which of the following sets are ideals in the given ring:

1. $\{p(x, y) \,|\, p(x, x) = 0\} \subset \mathbb{C}[x, y]$

2. $\{p(x,y) \mid p(x,y) = p(y,x)\} \subset \mathbb{C}[x,y]$
3. $\{p(x) \mid$ p has no real roots$\} \subset \mathbb{C}[x]$

Question 6

Let R be a commutative ring.

1. Prove that $Hom(\mathbb{Z}, R) := \{\phi : \mathbb{Z} \to R | \phi$ a homomorphism$\}$ contains one element.
2. Give an example of a commutative ring with unity R such that $Hom(R, \mathbb{Z}) = \varnothing$.
3. Give en example of a commutative ring with unity R such that $Hom(R, \mathbb{Z})$ is infinite.
4. Prove that the set $Hom(\mathbb{Z}[X], R)$ can be naturally put into bijection with the set R.

Question 7

Let R be a commutative ring with unity.

1. Let $(I_j)_{j \in A}$ be a family of ideals in R (ie, some arbitrary collection of ideals in R). Prove that $\bigcap_{j \in A} I_j$ is an ideal in R.
2. Let $X \subset R$ be an arbitrary subset. Prove that there exists a unique ideal $I \subset R$ containing X with the following property: if J is an ideal and $X \subset J$, then $I \subset J$. (*We call the ideal I just determined **the ideal generated by** X, and denote it $(X) \subset R$.*)
3. Determine (X) for the following subsets $X \subset R$

$$X = \{x-1, x+1\} \subset R = \mathbb{R}[x], \; X = \{x^2+1, x^2-1\} \subset R = \mathbb{C}[x].$$

Question 8

Is there an integral domain containing exactly 10 elements?

Question 9

Determine all automorphism of the ring $\mathbb{Z}[X]$. Hint: use question 6.

Question 10

Let R be a commutative ring. We say $r \in R$ is nilpotent if $r^n = 0_R$ for some $n \in \mathbb{N}$. Prove that the collection of nilpotent elements of a commutative ring form an ideal. Determine this ideal in the case $R = \mathbb{Z}/2700\mathbb{Z}$.