Math 113 Homework 2

To be submitted in class on Thursday July 5

Important Information:

- This homework will be graded for completeness and correctness. Two questions will be selected at random for close scrutiny. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make.
- Problems will be of varying difficulty, and do not appear in any order of difficulty. Expect to spend between 5 and 10 hours for each homework set.
- Good Luck!

Question 1

Let $(G, *)$ be a group. We define the center of G to be the subset:

$$Z(G) = \{g \in G | g * h = h * g \forall h \in G\}$$

Prove that it is a subgroup.

Question 2

Let $(G, *)$ be a group. Let $H, K \subset G$ be two subgroups. Prove that $H \cup K \subset G$ is a subgroup $\iff$ either $H \subset K$ or $K \subset H$.

Question 3

Prove that $gp([a]) = \mathbb{Z}/n\mathbb{Z}$ if and only if a is coprime to n.

Question 4

Let $(G, *)$ be a group and S a set. Let $\varphi : G \longrightarrow \Sigma(S)$ be a group homomorphism. Prove that the following map of sets:

$$\begin{array}{rcl} \mu : G \times S & \longrightarrow & S \\ (g, s) & \longrightarrow & \varphi(g)(s) \end{array}$$

is an action. This shows the equivalence between the two concepts of a group action.

Question 5

Let G be a finite group with 20 elements. Let S be a set with 15 elements. Does there exist a transitive action of G on S?

Question 6

Let p be a prime number. Prove that $(\mathbb{Z}/p\mathbb{Z}\backslash\{[0]\}, \times)$ is a group. Using this fact, prove Fermat's Little Theorem:

Given $a \in \mathbb{Z}$, coprime to p, then $a^{p-1} \equiv 1 \; mod(p)$.

Question 7

Let $(G, *)$ and $(H, \circ)$ be two groups. Assume $(G, *)$ is finitely generated with $gp(\{x_1, \cdot, x_n\}) = G$. Prove that if ϕ and ψ are two homomorphisms from G to H then

$$\phi(x_i) = \psi(x_i) \quad \forall x_i \quad \Rightarrow \quad \phi = \psi.$$

This proves the important fact that a homomorphism is completely determined by what it does to a generating set. [CAUTION: it does not follow, as in the case of linear algebra, that we can define a homomorphism simply by specifying where to send the generators; one has to be careful about possible relations the generators may satisfy]

Question 8

Let g be an element of order k in a group G.

a) If $\phi\colon G \to H$ is a homomorphism, prove that the order of $\phi(g)$ divides k.

b) If $\phi\colon G \to H$ is an isomorphism, prove that the order of $\phi(g)$ is equal to k.

Question 9

Determine all pairs $n, m \in \mathbb{N}$ such that there exists a non-trivial homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$. Hint: use the previous two problems.

Question 10

Does $[a](s) = a + s$ define an action of $(\mathbb{Z}/n\mathbb{Z}, +)$ on $\mathbb{R}$.

Question 11

Define an action of Sym_3 on $\mathbb{R}^3$ as follows: for $\sigma \in Sym_3$ and $v = (x_1, x_2, x_3) \in \mathbb{R}^3$, set

$$\sigma \cdot v = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}).$$

1. Show that the subspace $V = \{(x_1, x_2, x_3) \,|\, x_1 + x_2 + x_3 = 0\}$ has the following property: if $v \in V$, then $\sigma \cdot v$ is also in V.
2. Can you find a line (one-dimensional subspace) which has this same property?