MATH 113 MIDTERM EXAM
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

This exam consists of 7 questions. Answer the questions in the spaces provided.

1. (25 points) Let G be a set equipped with a binary operation $*$.

(a) Carefully define what it means for $(G, *)$ to be a group.
Solution:

$(G,*)$ is a group if the following properties are satisfied:

1/ $\forall x, y, z \in G$, $x*(y*z) = (x*y)*z$ (Associativity)

2/ $\exists e \in G$ such that $x*e = e*x = x$ $\forall x \in G$ (Identity)

3/ Given $x \in G$, $\exists y \in G$ such that $x*y = y*x = e$ (Inverses)

(b) Let $x \in G$. Prove that there is a unique element $y \in G$ such that $x * y = y * x = e$.
Solution:

Let $y, y' \in G$ such that $x*y = y*x = e = x*y' = y'*x$

$\Rightarrow$ $y' = y'*e = y'*(x*y) = (y'*x)*y = e*y = y$

2. (25 points) (a) Let G be a cyclic group such that $gp(\{x\}) = G$ and $|G| = n$. Prove the following:

$$gp(\{x^a\}) = G \iff HCF(a, n) = 1.$$

You may use any result from lectures as long as it is clearly stated.

Solution:

Given $y \in G$

$|gp(y)| = ord(y) = \min m \in \mathbb{N}$ such that $y^m = e$

$y^k = e \iff ord(y) \mid k$

$(\Rightarrow)$ $gp(\{x^a\}) = G \Rightarrow ord(x^a) = n \Rightarrow x^{ak} \neq e \ \forall k \in \{1, \dots, n-1\}$

$\Rightarrow n \nmid ak \ \forall k \in \{1, \dots n-1\} \Rightarrow HCF(a, n) = 1$

$(\Leftarrow)$

$HCF(a,n) = 1 \Rightarrow n \nmid ak \ \forall k \in \{1, \dots, n-1\} \Rightarrow x^{ak} \neq e \ \forall k \in \{1, \dots, n-1\}$

$\Rightarrow ord(x^a) \geq n$. $x^a \in G \Rightarrow ord(x^a) \mid |G| \Rightarrow ord(x^a) \mid n$

$\Rightarrow ord(x^a) = n \Rightarrow gp(\{x^a\}) = G$

(b) Is it possible for a group to have exactly 9 elements of order 4? Carefully justify your answer.

No

Both order 4 as 1, 3 coprime to 4

$ord(x) = 4 \Rightarrow gp(\{x\}) = \{e, x, x^2, x^3\} = gp(\{x^3\})$

So each cyclic subgroup contains exactly 2 elements of order 4.

Hence there must be an even number of such terms

3. (25 points) Let $(G, *)$ be a group together with an action on a set S

   (a) Prove that the orbits partition S. You may use any result from lectures are long as it is clearly stated.

   Solution:

- $e(x) = x \quad \forall x \in S \Rightarrow x \in Orb(x) \Rightarrow \bigcup_{x \in S} Orb(x) = S$
- Assume $Orb(x) \cap Orb(y) \neq \phi$

$\Rightarrow \exists\ g_1, g_2 \in G$ such that $g_1(x) = g_2(y)$

$\Rightarrow x = (g_1^{-1} g_2)(x)$ and $y = (g_2^{-1} g_1)(x)$

Let $g \in G$

$$\left.\begin{array}{l} g(x) = (g g_1^{-1} g_2)(y) \Rightarrow Orb(x) \subset Orb(y) \\ g(y) = (g g_2^{-1} g_1)(x) \Rightarrow Orb(y) \subset Orb(x) \end{array}\right\} \Rightarrow Orb(x) = Orb(y)$$

   (b) State, without proof, the orbit-stabilizer theorem

   Solution:

Let $x \in S$. The map $\phi : G/Stab(x) \longrightarrow Orb(x)$ is a
$$g\,Stab(x) \longrightarrow g(x)$$
well-defined bijection

   (c) Assume now that $|G| = 77$ and $|S| = 6$. Prove that the action is trivial.

   Solution:

Orbit-Stabilizer $\Rightarrow |G| = |Stab(x)| \cdot |Orb(x)| \quad \forall x \in S$

$|S| = 6 \Rightarrow |Orb(x)| \leq 6$, $|G| = 77 = 11 \times 7$ ← prime decomposition

$\Rightarrow |Orb(x)| = 1 \quad \forall x \in G \Rightarrow$ Action is trivial

4. (25 points) (a) Let G and H be two groups. Define what it means for $\phi : G \to H$ to be a homomorphism.

Solution:

in G   in H

$\phi : G \longrightarrow H$ is a homomorphism iff $\phi(x * y) = \phi(x) \circ \phi(y)$

$\forall\, x, y \in G$

(b) State and prove the first isomorphism theorem for groups. You may use any result from lectures are long as it is clearly stated.

Solution:

Let $\phi : G \longrightarrow H$ be a homomorphism. Then the map $\psi : G/\mathrm{Ker}\,\phi \to \mathrm{Im}\,\phi$

$x\,\mathrm{Ker}\,\phi \to \phi(x)$

is a well defined isomorphism.

Proof

$x\,\mathrm{Ker}\,\phi = y\,\mathrm{Ker}\,\phi \iff x^{-1}y \in \mathrm{Ker}\,\phi \iff \phi(x^{-1}y) = e_H \iff (\phi(x))^{-1}\phi(y) = e_H$

$\iff \phi(x) = \phi(y) \implies \psi$ both well defined and injective.

ψ surjective by definition of $\mathrm{Im}\,\phi$

$\psi((x\,\mathrm{Ker}\,\phi)(y\,\mathrm{Ker}\,\phi)) = \psi(xy\,\mathrm{Ker}\,\phi) = \phi(xy) = \phi(x)\phi(y)$

$= \psi(x\,\mathrm{Ker}\,\phi)\,\psi(y\,\mathrm{Ker}\,\phi)$

$\square$

5. (25 points) (a) Let $\sigma \in Sym_n$. Define what it means for $\sigma \in Alt_n$. Is the permutation (123)(345)(5267) contained in Alt_7?

Solution:

$\sigma \in Alt_n \iff$ It has an even number of even length cycles in its cycle structure

One even length cycle

$(123)(345)(5267) = (1267)(345) \notin Alt_7$

(b) Determine the center $Z(Sym_5) \subset Sym_5$. Hint: Consider conjugacy classes. You may use any result from lectures are long as it is clearly stated.

Solution:

$\sigma \in Z(Sym_5) \iff Conj(\sigma) = \{\tau\sigma\tau^{-1} \mid \tau \in Sym_5\} = \{\sigma\}$

$Conj(\sigma)$ = All permutations with same cycle structure as σ

Possible cycle structures:

5	$(12345) \neq (13245)$
4, 1	$(1234) \neq (1324)$
3, 1, 1	$(123) \neq (132)$
3, 2	$(123)(45) \neq (132)(45)$
2, 2, 1	$(12)(34) \neq (13)(45)$
2, 1, 1, 1	$(12) \neq (23)$
1, 1, 1, 1, 1	e

$\Rightarrow Z(Sym_5) = \{e\}$

6. (25 points) (a) Let G be a group and $N \subset G$ a subgroup. Define what it means for N to be normal. Is it true that G/N Abelian $\Rightarrow G$ is Abelian? Carefully justify your answer. You may use any result from lectures are long as it is clearly stated.

Solution:

$N \triangleleft G \iff n \in N, g \in G \Rightarrow gng^{-1} \in N$

G/N Abelian $\not\Rightarrow$ G Abelian

union of conjugacy classes so normal

For example, $G = Sym_3$, $N = \{e, (123), (132)\}$

$|G/N| = |G|/|N| = 3!/3 = 2$ prime $\Rightarrow G/N \cong \mathbb{Z}/2\mathbb{Z} \Rightarrow G/N$ Abelian

However $G = Sym_3$ is non-Abelian

(b) Give an example of a group G and a subgroup $N \subset G$ such that N is **not** normal. Carefully justify your answer.

You may use any result from lectures are long as it is clearly stated.

Solution:

$G = Sym_3$, $N = \{e, (12)\}$

$N \not\triangleleft G$ as N is not the union of conjugacy classes.

$Conj(12) = \{(12), (23), (13)\}$ cycle structure $\{2,1\}$

7. (25 points) (a) Let G be a finite group. State, without proof, Sylow's Theorem.
Solution:

Let G be a finite group and p a prime.

$p^n \mid |G| \Rightarrow \exists$ a subgroup $H \subset G$ such that $|H| = p^n$.

(b) Let p be a prime. Prove the following:
p divides $|G| \Rightarrow$ There exists an element of order p in G.
Solution:

By Sylow $\exists$ $H \subset G$ such that $|H| = p$

Let $x \in H$, $x \neq e$ $\Rightarrow$ $\mathrm{ord}(x) > 1$ and $\mathrm{ord}(x) \mid p$

$\Rightarrow$ $\mathrm{ord}(x) = p$.

(c) Again for p prime, is the following statement true?
p^n divides $|G| \Rightarrow$ There exists an element of order p^n in G.
Carefully justify your answer.

It is not true. For example $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

$p^2 \mid |G|$, however $\mathrm{ord}(x) \leq p$ $\forall x \in G$.

END OF EXAM