# MATH 113 MIDTERM EXAM 4.10PM-6PM
# PROFESSOR PAULIN

**DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.**

**CALCULATORS ARE NOT PERMITTED**

**REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT**

**THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM**

Name: ________________________________________

**This exam consists of 7 questions. Answer the questions in the spaces provided.**

1. (25 points) (a) Carefully define what it means for a set $G$ to be a group.
**Solution:**

(b) Prove that the identity is unique in a group $G$.
**Solution:**

(c) Let $G$ be a group and $H \subset G$. Define what it means for $H$ to be a subgroup.

**Solution:**

(d) Let $H \subset G$ be a subgroup. Prove the following is an equivalence relation:

$$x \sim y \iff x^{-1}y \in H$$

**Solution:**

PLEASE TURN OVER

2. (25 points) Let $G$ be a group.

   (a) Define what it means for a subgroup $N \subset G$ to be normal.
   **Solution:**

   (b) If $N \subset G$ is a normal subgroup, prove that the binary operation

$$\begin{aligned} \phi : G/N \times G/N &\longrightarrow G/N \\ (xN, yN) &\longrightarrow (xy)N \end{aligned}$$

   is well-defined, i.e. independent of coset representative choices.
   **Solution:**

   (c) Prove that $G$ cyclic $\Rightarrow G/N$ cyclic
   **Solution:**

3. (25 points) (a) State, without proof, Lagrange's Theorem.

**Solution:**

(b) Let $G$ be a group and $x, y \in G$ such that $ord(x)$ and $ord(y)$ are coprime. Prove that if $n, m \in \mathbb{Z}$ then

$$x^n = y^m \Rightarrow ord(x)|n \text{ and } ord(y)|m$$

You may use any result from lectures as long as it is clearly stated.

4. (25 points) Let $G$ be a group and $S$ be a set.

   (a) Define the concept of an action of $G$ in $S$.

   **Solution:**

   (b) Prove that

$$\begin{aligned} \phi : G \times G &\longrightarrow G \\ (g,h) &\longrightarrow ghg^{-1} \end{aligned}$$

   gives a group action on $G$ on itself.

   **Solution:**

   (c) Using this, prove the following: If $G$ is finite then

$$|\{ghg^{-1} | g \in G\}| \text{ divides } |G| \text{ for any } h \in G.$$

   You may use any result from the course as long as it is clearly stated.

5. (25 points) Show that for $x, y \in Sym_5$, if $ord(x) = ord(y) = 6$, then $x$ and $y$ are conjugate. Is the same true of elements of order 2? You may use any result from the course as long as it is clearly stated.

   **Solution:**

6. (25 points) Let $G$ and $H$ be groups.

   (a) Define the concept of a homomorphism from $G$ to $H$.

   **Solution:**

   (b) State, without proof, the first isomorphism theorem for groups.

   **Solution:**

   (c) Give an example of a non-trivial homomorphism from $\mathbb{Z}/3\mathbb{Z}$ to $D_6$. You do not need to prove it is a homomorphism.

   **Solution:**

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7. (25 points) (a) State the structure theorem for finitely generated Abelian groups.

**Solution:**

(b) Using this, show that an Abelian group of order 30 must contain an element of order 5.

**Solution:**

(c) Prove that, up to isomorphism, there is only one group of size 100, such that every element has order dividing 10.

END OF EXAM