MATH 113 PRACTICE MIDTERM EXAM
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

This exam consists of 7 questions. Answer the questions in the spaces provided.

1. (25 points) Let $(G, *)$ be a group.
   (a) Let $H \subset G$. Definite what it means for H to be a subgroup of G.
   Solution:

1/ $e \in H$

2/ $h \in H \iff h^{-1} \in H$

3 $g, h \in H \Rightarrow gh \in H$

   (b) For $x \in G$, let $xH = \{x * h | h \in H\}$. Prove that $y \in xH \iff yH = xH$.
   Solution:

$(\Rightarrow)$ $y \in xH \Rightarrow y = xh_1$ for some $h_1 \in H$

$\Rightarrow$ $yh = x(\overbrace{h_1 h}^{\in H})$ $\forall h \in H \Rightarrow yH \subset xH$

$y = xh_1 \Rightarrow yh_1^{-1} = x \Rightarrow y(\underbrace{h_1^{-1}h}_{\in H}) = xh$ $\forall h \in H$

$\Rightarrow xH \subset yH \Rightarrow yH = xH.$

$(\Leftarrow)$ $e \in H \Rightarrow y \in yH. \Rightarrow y \in xH.$

(c) Show that if H is of finite index in G, then there exist $x_1, \cdots, x_n \in G$ such that given any $x \in G$, $x = x_i * h$ for some x_i and some $h \in H$

Solution:

$|G/H| < \infty \Rightarrow G/H = \{x_1 H, \ldots, x_n H\}$ for some finite set $\{x_1, \ldots, x_n\} \subset G$. The cosets of H in G form a partition of G. Hence given $x \in G$

$x \in x_i H$ for some $i \in \{1, \ldots, n\}$.

$\Rightarrow x = x_i h$ where $h \in H$.

2. (25 points) Let H and G be two groups.

(a) What is a homomorphism ϕ from G to H?

Solution: A homomorphism from G to H is a map $\phi: G \to H$ such that $\phi(\underbrace{xy}_{\text{composition in } G}) = \underbrace{\phi(x)\,\phi(y)}_{\text{composition in } H.}$ $\forall x, y \in G$

(b) Define the $ker\phi \subset G$. Prove that it is a normal subgroup. You may assume any standard results about homomorphisms from lectures.

Solution:

$\ker\phi = \{ g \in G \mid \phi(g) = e_H \}$

1/ $\phi(e_G) = e_H \Rightarrow e_G \in \ker\phi$

2/ $\phi(x) = e_H \Rightarrow (\phi(x))^{-1} = e_H \Rightarrow \phi(x^{-1}) = e_H \Rightarrow x^{-1} \in \ker\phi$

3/ $x, y \in \ker\phi \Rightarrow \phi(xy) = \phi(x)\phi(y) = e_H e_H = e_H \Rightarrow xy \in \ker\phi$

4/ $x \in \ker\phi, y \in G \Rightarrow \phi(yxy^{-1}) = \phi(y)\phi(x)\phi(y)^{-1} = \phi(y)\phi(y)^{-1} = e_H$
$\Rightarrow yxy^{-1} \in \ker\phi$

$\Rightarrow \ker\phi \triangleleft G$.

(c) State, without proof, the First Isomorphism Theorem.

Solution: Let $\phi: G \to H$ be a homomorphism then

$\psi: G/\ker\phi \to \mathrm{Im}\,\phi$, $x\ker\phi \to \phi(x)$ is a well-defined isomorphism.

(d) Using this, or otherwise, show that there are no non-trivial homomorphisms from $\mathbb{Z}/5\mathbb{Z}$ to D_{11}.

$\phi: \mathbb{Z}/5\mathbb{Z} \to D_{11}$ a homomorphism

Solution:

1st Isomorphism Theorem $\Rightarrow |\mathrm{Im}\,\phi| \,/\, 5$

Lagrange $\Rightarrow |\mathrm{Im}\,\phi| \,/\, 22$

$HCF(22, 5) = 1 \Rightarrow |\mathrm{Im}\,\phi| = 1 \Rightarrow \phi$ trivial.

3. (25 points) Let G be a group.

   (a) If $x \in G$ is of finite order, define $ord(x)$. You need only give one of the two equivalent definitions.

   Solution:

$ord(x) =$ minimal $m \in \mathbb{N}$ s.t. $x^m = e$

$(ord(x) = |gp(\{x\})|)$

   (b) Prove that if $d \in \mathbb{N}$ such that $x^d = e$, then $ord(x)|d$.

   Solution:

Assume $x^d = e$ and $m \nmid d$ $(ord(x) = m)$

$\Rightarrow$ $d = qm + r$ where $0 < r < m$ $\Rightarrow$

$e = x^d = x^{qm+r} = (x^m)^q \cdot x^r = x^r$.

This is a contradiction by the minimality of m.

$\Rightarrow$ $ord(x) \mid d$

   (c) If $|G| = 20$, is it possible that there is $x \in G$ such that $ord(x) = 3$. You may use any result from lectures as long as it is clearly stated.

Lagrange $\Rightarrow$ $ord(x) = |gp(\{x\})| \,\big|\, |G|$

$3 \nmid 20$ $\Rightarrow$ If $|G| = 20$ $\nexists\, x \in G$ s.t.

$ord(x) = 3$.

4. (25 points) Let S be a set equipped with an action of a group G.

(a) Define what it means for the action to be transitive. Be sure to carefully explain any terminology you use.

Solution:

The action is transitive if $\mathrm{orb}(s) = \{g(s) \mid g \in G\} = S$ $\forall s \in S$.

(b) Define what it means for the action to be faithful.

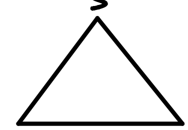
Solution:

The action is faithful if the induced homomorphism $\varphi : G \to \Sigma(S)$ is injective.

(c) Give an example of an action which is both faithful and transitive. Give an example of an action that is transitive but not faithful.

Solution:

S = vertices of an equilateral triangle



$G_1 = \mathbb{Z}/3\mathbb{Z}$. Let $[a]_3$ act on $\{1,2,3\}$ by rotation anticlockwise by $\frac{2\pi}{3}a$. This action is both transitive and faithful.

$G_2 = \mathbb{Z}/6\mathbb{Z}$. Let $[a]_6$ act on $\{1,2,3\}$ by rotation anticlockwise by $\frac{2\pi}{3}a$. This action is transitive but not faithful. E.g.

$$[3]_6(1) = (1) = [0]_6(1)$$
$$[3]_6(2) = (2) = [0]_6(2)$$
$$[3]_6(3) = (3) = [0]_6(3)$$

5. (25 points) (a) Define what it means for a subgroup $N \subset G$ to be normal. Define what it means for G to be simple.

Solution:

$N \lhd G \Rightarrow N$ is a subgroup and $gng^{-1} \in N \quad \forall g \in G, n \in N$

G is simple if $N \lhd G \Rightarrow N = \{e\}$ or $N = G$.

(b) State, without proof, the Third Isomorphism Theorem.

Solution:

Let G be a group and $N \lhd G$.

1/ There is an inclusion preserving bijection between $\left\{ \begin{matrix} \text{subgroups of} \\ G/N \end{matrix} \right\}$ and $\left\{ \begin{matrix} \text{Subgroups of} \\ G \text{ containing } N \end{matrix} \right\}$.

$$H/N \longleftrightarrow H$$

2/ $H/N \lhd G \Leftrightarrow H \lhd G$ and in this case $(G/N)/(H/N) \cong G/H$.

(c) Using this, prove that if there are no normal subgroups of G strictly between G and N, then G/N is simple.

Solution:

If G/N is not simple then exists $N < H < G$ s.t.

$H/N \lhd G/N$ and $H/N \neq \{eN\}$ or G/N

$\Rightarrow N \subsetneq H \subsetneq G$

$H/N \lhd G/N \Rightarrow H \lhd G$. This is a contradiction, hence G/N is simple.

6. (25 points) (a) How many conjugacy classes of Sym_5 are there. Give an example of three elements, none of which are conjugate.

Solution:

Number of conjugacy classes of Sym_5 = Number of partitions of 5

$1+1+1+1+1$
$1+1+1+2$
$1+2+2$
$1+1+3$
$2+3$
$1+4$
5

$\Rightarrow$ There are 7 conjugacy classes of Sym_5

$(12), (123), (1234)$ are 3 non-conjugate elements.

(b) What is the highest possible order of an element in Sym_5. Using this, or otherwise, prove that Sym_5 is not cyclic.

Solution:

$1,1,1,1,1 \rightsquigarrow LCM = 1$
$1,1,1,2 \rightsquigarrow LCM = 2$
$1,2,2 \rightsquigarrow LCM = 2$
$1,1,3 \rightsquigarrow LCM = 3$
$2,3 \rightsquigarrow LCM = 6$
$1,4 \rightsquigarrow LCM = 4$
$5 \rightsquigarrow LCM = 5$

$\Rightarrow$ Max order of $x \in Sym_5$ is 6. $|Sym_5| = 5! = 120$

$\Rightarrow \nexists\, x \in Sym_5$ s.t. $gp(\{x\}) = Sym_5$.

$\Rightarrow Sym_5$ is <u>not</u> cyclic.

7. (25 points) Let G be a finitely generated Abelian group.

(a) Define the torsion subgroup $tG \subset G$. Prove that it is a subgroup.

Solution:

$$tG = \{x \in G \mid \mathrm{ord}(x) < \infty\}$$

1/ $\mathrm{ord}(0) = 1 \Rightarrow 0 \in tG$

2/ $x \in tG \Rightarrow$ $nx = 0$ for $n \in \mathbb{N} \Rightarrow -(nx) = 0 \Rightarrow n(-x) = 0 \Rightarrow -x \in tG$

3/ $x, y \in tG \Rightarrow nx = 0$ and $my = 0$ for some $n, m \in \mathbb{N}$.

$\Rightarrow (nm)(x+y) = m(nx) + n(my) = 0 + 0 = 0 \Rightarrow x+y \in tG$.

(b) Prove that G/tG is torsion-free.

Solution:

Let $x + tG \in t(G/tG) \Rightarrow \exists\, n \in \mathbb{N}$ s.t. $n(x + tG)$

$= 0 + tG \Rightarrow nx + tG = 0 + tG \Rightarrow nx \in tG$

$\Rightarrow \exists\, m \in \mathbb{N}$ $m(nx) = (mn)x = 0 \Rightarrow x \in tG$

$\Rightarrow x + tG = 0 + tG \Rightarrow t(G/tG) = \{0 + tG\}$.

(c) Give an example of a torsion group that is infinite. Make sure you justify why it is torsion.

$(\mathbb{Q}/\mathbb{Z}, +)$. $[\frac{a}{b}] \in \mathbb{Q}/\mathbb{Z} \Rightarrow b[\frac{a}{b}] = [a] = [0]$

$\Rightarrow \{\frac{a}{b}\} = t(\mathbb{Q}/\mathbb{Z})$.

END OF EXAM