MATH 113 PRACTICE MIDTERM EXAM
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

This exam consists of 7 questions. Answer the questions in the spaces provided.

1. (25 points) Let $(G, *)$ be a group.

   (a) Let $H \subset G$. Definite what it means for H to be a subgroup of G.

   Solution:

   (b) For $x \in G$, let $xH = \{x * h | h \in H\}$. Prove that $y \in xH \iff yH = xH$.

   Solution:

(c) Show that if H is of finite index in G, then there exist $x_1, \cdots, x_n \in G$ such that given any $x \in G$, $x = x_i * h$ for some x_i and some $h \in H$

Solution:

2. (25 points) Let H and G be two groups.

   (a) What is a homomorphism ϕ from G to H?

   Solution:

   (b) Define the $ker\phi \subset G$. Prove that it is a normal subgroup. You may assume any standard results about homomorphisms from lectures.

   Solution:

   (c) State, without proof, the First Isomorphism Theorem.

   Solution:

   (d) Using this, or otherwise, show that there are no non-trivial homomorphisms from $\mathbb{Z}/5\mathbb{Z}$ to D_{11}.

   Solution:

PLEASE TURN OVER

3. (25 points) Let G be a group.

   (a) If $x \in G$ is of finite order, define $ord(x)$. You need only give one of the two equivalent definitions.

   Solution:

   (b) Prove that if $d \in \mathbb{N}$ such that $x^d = e$, then $ord(x)|d$.

   Solution:

   (c) If $|G| = 20$, is it possible that there is $x \in G$ such that $ord(x) = 3$. You may use any result from lectures as long as it is clearly stated.

4. (25 points) Let S be a set equipped with an action of a group G.

   (a) Define what it means for the action to be transitive. Be sure to carefully explain any terminology you use.

   Solution:

   (b) Define what it means for the action to be faithful.

   Solution:

   (c) Give an example of an action which is both faithful and transitive. Give an example of an action that is transitive but not faithful.

   Solution:

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5. (25 points) (a) Define what it means for a subgroup $N \subset G$ to be normal. Define what it means for G to be simple.

   Solution:

   (b) State, without proof, the Third Isomorphism Theorem.

   Solution:

   (c) Using this, prove that if there are no normal subgroups of G strictly between G and N, then G/N is simple.

   Solution:

PLEASE TURN OVER

6. (25 points) (a) How many conjugacy classes of Sym_5 are there. Give an example of three elements, none of which are conjugate.

   Solution:

   (b) What is the highest possible order of an element in Sym_5. Using this, or otherwise, prove that Sym_5 is not cyclic.

   Solution:

7. (25 points) Let G be a finitely generated Abelian group.

   (a) Define the torsion subgroup $tG \subset G$. Prove that it is a subgroup.
   Solution:

   (b) Prove that G/tG is torsion-free.
   Solution:

   (c) Give an example of a torsion group that is infinite. Make sure you justify why it is torsion.

END OF EXAM