# MATH 113 PRACTICE MIDTERM EXAM
## PROFESSOR PAULIN

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| DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO. |
| CALCULATORS ARE NOT PERMITTED |
| REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT |
| THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM |

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**This exam consists of 7 questions. Answer the questions in the spaces provided.**

1. (25 points) Let $G$ be a set.
   (a) What is a binary operation on $G$?
   **Solution:**

A binary operation on $G$ is a map of sets

$$* : G \times G \to G$$
$$(g, h) \mapsto g * h$$

   (b) Carefully define what it means for a set $G$ with a binary operation $*$ to be a group.
   **Solution:**

Let $G$ be a set equipped with a binary operation $*$. $(G, *)$ is a group if the following are satisfied:

1. $(g*h)*x = g*(h*x) \quad \forall g, h, x \in G$
2. $\exists e \in G$ s.t. $g*e = e*g = g \quad \forall g \in G$
3. Given $g \in G$, $\exists h \in G$ s.t. $h*g = g*h = e$

(c) Let $(G, *)$ be a group and $g \in G$. Prove that the map

$$\begin{aligned} \phi_g : G &\to G \\ h &\to g^{-1} * h * g \end{aligned}$$

is an automorphism. Carefully justify your answer

**Solution:**

Claim : $\phi_g$ is a homomorphism.

Proof : Let $x, y \in G$. Then

$$\phi_g(xy) = g^{-1}(xy)g = (g^{-1}xg)(g^{-1}yg)$$

$$= \phi_g(x)\phi_g(y).$$

$\square$

Claim : $\phi_g$ is a bijection.

Proof : Consider $\phi_{g^{-1}} : G \to G$, $h \mapsto ghg^{-1}$

$$\Rightarrow (\phi_g \circ \phi_{g^{-1}})(h) = \phi_g(ghg^{-1}) = g^{-1}(ghg^{-1})g = h$$

$\forall h \in G \Rightarrow \phi_g \circ \phi_{g^{-1}} = Id_G$

$$(\phi_{g^{-1}} \circ \phi_g)(h) = \phi_{g^{-1}}(g^{-1}hg) = g(g^{-1}hg)g^{-1} = h$$

$\forall h \in G \Rightarrow \phi_{g^{-1}} \circ \phi_g = Id_G$

$\Rightarrow \phi_g$ is a bijection.

$\square$

2. (25 points) Let $(G, *)$ be a group and $S$ a set.

(a) What is an *action* of $G$ on $S$?

**Solution:**

An action of $G$ on $S$ is a function

$\mu : G \times S \to S$ such that
$(g, s) \mapsto g(s)$

1/ $e(s) = s \ \forall s \in S$ and 2/ $(g * h)(s) = g(h(s)) \ \forall g, h \in G, s \in S$.

(b) Assume we are given an action $\varphi$, of $G$ on $S$. Let $s \in S$. Define $stab(s) \subset G$ and $orb(s) \subset S$.

**Solution:**

$Stab(s) = \{ g \in G \mid g(s) = s \} \subset G$

$orb(s) = \{ g(s) \mid g \in G \} \subset S$

(c) State, without proof, the orbit-stabilizer theorem

**Solution:**

Let $G$ be a finite group acting on a set $S$.

Then $|G| = |Stab(s)| \cdot |orb(s)| \ \forall s \in S$.

(d) If $|G| = 5$ is it possible for there to be an action of $G$ on a set of size 5, where there are precisely 2 orbits?

**Solution:**

Orbit-Stabiliser $\Rightarrow$ $|orb(s)| \mid 5 \ \forall s \in S$

Possible orbit sizes :

$1, 4 \leftarrow 4 \nmid 5$
$2, 3 \leftarrow 3 \nmid 5$

$\Rightarrow$ Neither can occur from an action of $|G|$ on a set of size 5 with 2 orbits

$\Rightarrow$ There is no such action.

3. (25 points) (a) State, without proof, Lagrange's Theorem.

**Solution:**

Let $G$ be a finite group and $H \subset G$ a subgroup.

Then $|H| \,/\, |G|$.

(b) Using this prove that all groups of prime order are simple. Is the same true of all groups of prime power order? Carefully justify your answers.

**Solution:**

Let $H \subset G$ be a subgroup and $|G| = p$, a prime.

$$\Rightarrow |H| \,\big|\, p \Rightarrow \begin{array}{c} |H| = 1 \\ \text{or} \\ |H| = p \end{array} \Rightarrow \begin{array}{c} H = \{e\} \\ \text{or} \\ H = G \end{array}$$

Thus $G$ contains only 2 subgroups, $\{e\}$ and $G$.

Hence $G$ is simple.

The same is not true for all groups of prime power order. For example $G = (\mathbb{Z}/p^2\mathbb{Z}, +)$

$\operatorname{ord}([p]) = p \Rightarrow gp(\{[p]\}) \subset G$ is a non-trivial subgroup. $\mathbb{Z}/p^2\mathbb{Z}$ is Abelian, hence $gp(\{[p]\}) \triangleleft \mathbb{Z}/p^2\mathbb{Z} \Rightarrow \mathbb{Z}/p^2\mathbb{Z}$ <u>not</u> simple.

4. (25 points) (a) Define what it means for a group to be cyclic.

**Solution:**

$G$ is cyclic $\iff \exists\, x \in G$ s.t. $gp(\{x\}) = G$.

(b) Prove that if $G$ is cyclic and $|G| = n \in \mathbb{N}$, then $G \cong \mathbb{Z}/n\mathbb{Z}$. You may assume any result from lectures are long as it is clearly stated.

**Solution:**

Assume $G = gp(\{x\})$ and $|G| = n$.

Claim : The map $\phi : \mathbb{Z}/n\mathbb{Z} \to G$
$[a] \to x^a$

is a well-defined isomorphism.

Proof $|G| = n \Rightarrow ord(x) = n \Rightarrow x^k = e \iff n \mid k$.

$[a] = [b] \Rightarrow n \mid a-b \Rightarrow x^{a-b} = e \Rightarrow x^a = x^b$

$\Rightarrow \phi$ well defined.

$\phi([a]+[b]) = \phi([a+b]) = x^{a+b} = x^a \cdot x^b = \phi([a])\,\phi([b])$

$\Rightarrow \phi$ is a homomorphism.

$gp(\{x\}) = G \Rightarrow \phi$ surjective.

$|G| = |\mathbb{Z}/n\mathbb{Z}| \Rightarrow \phi$ bijective.

$\square$

5. (a) (20 points) Determine the number of cyclic subgroups of order 3 contained in $Sym_5$.
**Solution:**

$ord(x) = 3 \iff x$ has cycle structure $3, 1, 1$.

Hence we must first determine the number of cycles of length 3, $(abc)$. $\left((abc) = (bca) = (cab)\right)$

$\Rightarrow$ There are $\frac{5 \times 4 \times 3}{3} = 20$ such cycles.

$gp(\{(abc)\}) = \{e, (abc), (acb)\}$

$\Rightarrow$ There are $\frac{20}{2} = 10$ cyclic subgroups of order 3 in $Sym_5$.

(b) (5 points) Prove that none of these are normal in $Sym_5$. You may use any result from lectures as long as it is clearly stated. Is $Sym_5$ a simple group?
**Solution:**

Recall that $\sigma, \tau \in Sym_5$ are conjugate $\iff$ $\sigma$ and $\tau$ have the same cycle structure

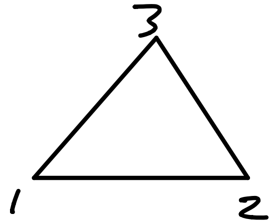
Hence if $(abc) \in N$ and $N \triangleleft Sym_5$ then $N$ must contain all cycles with this structure. There are 20 such cycles. Hence $|N| \geqslant 20$.

Hence none of these cyclic subgroups are normal in $Sym_5$.

6. (25 points) (a) Define the dihedral group $D_3$.

**Solution:**

$D_3 = Sym$ ( Equilateral Triangle )




(b) Prove that $D_3 \cong Sym_3$. In general is it true that $D_n \cong Sym_n$? Carefully justify your answer.

**Solution:**

There is a natural action of $D_3$ on $\{1, 2, 3\}$, hence a homomorphism $\phi : D_3 \longrightarrow Sym_3$.

Any $\sigma \in D_3$ is completely determined by what it does to $\{1, 2, 3\}$. Hence the action is faithful and $\phi$ is injective. $|D_3| = 2 \cdot 3 = 6 = 3! = |Sym_3|$

$\Rightarrow$ $\phi$ bijective $\Rightarrow$ $\phi$ is an isomorphism.

For $n > 3$ $|D_n| = 2n \neq n! = |Sym_n|$

$\Rightarrow$ $D_n \not\cong Sym_n$ for $n > 3$.

7. (25 points) (a) State the Structure Theorem for Finitely Generated Abelian Groups.

**Solution:**

A finitely generated Abelian group is isomorphic to the direct product of finitely many cyclic groups. These group are either infinite or prime power order. Such a decomposition is unique up to reordering and isomorphism.

(b) Let

$$G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}.$$

What is the rank of $G$? Explicitly describe the torsion subgroup of $G$ and prove that it is cyclic.

**Solution:**

$\text{Rank}(G) = 2$, $tG = \{(0,0,[a]_{25},[b]_9) \mid a,b \in \mathbb{Z}\}$

$tG \cong \mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$

25 and 9 coprime

$\Downarrow$

and $(([1]_{25},[1]_9)) = \text{LCM}(25,9) = 25 \times 9 = |tG|$.

$\Rightarrow gp(\{([1]_{25},[1]_9)\}) = tG \Rightarrow tG$ cyclic.

(c) Up to isomorphism, how many Abelian groups are there of order 16 are there? Is this all possible groups of order 16? Hint: Consider $D_4$.

$16 = 2^4$.

$$\left.\begin{array}{l} 1+1+1+1 \\ 1+1+2 \\ 2+2 \\ 3+1 \\ 4 \end{array}\right\} \text{There are 5 partitions of 4}$$

$\Rightarrow$ Up to isomorphism there a 5 Abelian groups of order 16.

$D_4$ is non-Abelian and $|D_4| = 8$

$\Rightarrow D_4 \times \mathbb{Z}/2\mathbb{Z}$ is non-Abelian and is size 16.

END OF EXAM