MATH 113 PRACTICE MIDTERM EXAM
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

This exam consists of 7 questions. Answer the questions in the spaces provided.

1. (25 points) Let G be a set.
   (a) What is a binary operation on G?
   Solution:

   (b) Carefully define what it means for a set G with a binary operation $*$ to be a group.
   Solution:

(c) Let $(G, *)$ be a group and $g \in G$. Prove that the map

$$\begin{aligned} \phi_g : G &\rightarrow G \\ h &\rightarrow g^{-1} * h * g \end{aligned}$$

is an automorphism. Carefully justify your answer

Solution:

2. (25 points) Let $(G, *)$ be a group and S a set.

   (a) What is an *action* of G on S?

   Solution:

   (b) Assume we are given an action φ, of G on S. Let $s \in S$. Define $stab(s) \subset G$ and $orb(s) \subset S$.

   Solution:

   (c) State, without proof, the orbit-stabilizer theorem

   Solution:

   (d) If $|G| = 5$ is it possible for there to be an action of G on a set of size 5, where there are precisely 2 orbits?

   Solution:

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3. (25 points) (a) State, without proof, Lagrange's Theorem.

Solution:

(b) Using this prove that all groups of prime order are simple. Is the same true of all groups of prime power order? Carefully justify your answers.

Solution:

4. (25 points) (a) Define what it means for a group to be cyclic.

   Solution:

   (b) Prove that if G is cyclic and $|G| = n \in \mathbb{N}$, then $G \cong \mathbb{Z}/n\mathbb{Z}$. You may assume any result from lectures are long as it is clearly stated.

   Solution:

5. (a) (20 points) Determine the number of cyclic subgroups of order 3 contained in Sym_5.
Solution:

(b) (5 points) Prove that none of these are normal in Sym_5. You may use any result from lectures as long as it is clearly stated. Is Sym_5 a simple group?
Solution:

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6. (25 points) (a) Define the dihedral group D_3.

Solution:

(b) Prove that $D_3 \cong Sym_3$. In general is it true that $D_n \cong Sym_n$? Carefully justify your answer.

Solution:

7. (25 points) (a) State the Structure Theorem for Finitely Generated Abelian Groups.

Solution:

(b) Let

$$G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}.$$

What is the rank of G? Explicitly describe the torsion subgroup of G and prove that it is cyclic.

Solution:

(c) Up to isomorphism, how many Abelian groups are there of order 16 are there? Is this all possible groups of order 16? Hint: Consider D_4.

END OF EXAM