MATH 113 FINAL EXAM (4.10PM-6PM)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ____

This exam consists of 7 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Carefully define what it means for a set R to be a field. State all the axioms precisely. Give two examples of a field, neither of which is contained in the other.

   Solution:

(b) Prove that a field is an integral domain. If you use any result from lectures be sure to state it clearly.

Solution:

(c) Is the converse to b) true? Be sure to justify your answer.

Solution:

2. (25 points) Let R and S be non-trivial rings and $\phi : R \to S$ be a ring homomorphism.

   (a) Define $ker(\phi) \subset R$ and prove it is an ideal. You do not need to prove it is a subgroup under addition.

   Solution:

   (b) Prove that $ker(\phi) \subset R$ is not a subring. You may assume any results from the lectures as long as they are clearly stated.

   Solution:

   (c) Prove that if R is a field then R is isomorphic to a subring of S. You may assume any results from the lectures as long as they are clearly stated

   Solution:

3. (25 points) Let R be a commutative ring

   (a) Define what it means for an ideal $I \subset R$ to be prime.

   Solution:

   (b) Prove that R/I is an integral domain if and only if I is prime.

   Solution:

   (c) Give an example of a prime ideal which is not maximal.

   Solution:

4. (25 points) Let R be an integral domain.

   (a) Define what is means for R to be UFD.

   Solution:

   (b) Define what it means for $r \in R$ to be prime. Prove that r prime $\Rightarrow$ r irreducible.

   Solution:

   (c) Prove that in a UFD, r irreducible $\Rightarrow$ r prime.

5. (25 points) Let F be a field, $\alpha \in F$ and $f(x) \in F[x]$ such that $deg(f(x)) \geq 1$. Prove $f(\alpha) = 0_F \iff (x - \alpha)|f(x)$ in $F[x]$. You may assume any results from the lectures as long as they are clearly stated. Is it true that if $f(x)$ is reducible in $F[x]$ then there exists $\alpha \in F$ such that $f(\alpha) = 0_F$.?

Solution:

6. (25 points) (a) Define what it means for $f(x) \in \mathbb{Z}[x]$ to be primitive.

Solution:

(b) State, without proof, Gauss' Lemma for polynomials in $\mathbb{Z}[x]$.

Solution:

(c) Does the polynomial $f(x) = 2x^{11} - 98x^5 + 28x^2 + 35$ have any roots in $\mathbb{Z}$? Is the ring $\mathbb{Q}[x]/(f(x))$ a field? Is the ring $\mathbb{C}[x]/(f(x))$ a field? You may assume any results from the lectures as long as they are clearly stated.

Solution:

7. (25 points) (a) Let E/F be a field extension. Define what if means for $\alpha \in E$ to be algebraic over F.

   Solution:

   (b) Prove that $\sqrt{2} + \sqrt{3}$ is algebraic over $\mathbb{Q}$.

   Solution:

(c) Using this, or otherwise, prove that if $\alpha \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$, then there exists $f(x) \in \mathbb{Q}[x]$ non-constant, such that $deg(f(x) \leq 4$ and $f(\alpha) = 0$. You may assume any results from the lectures as long as they are clearly stated.

Solution:

END OF EXAM