MATH 113 FINAL EXAM (PRACTICE 2)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

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This exam consists of 7 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Let R be a ring. Define what it means for a subset $S \subset R$ to be a subring. State all the axioms precisely.

Solution:

Let R be a ring. $S \subset R$ is a subring if

1/ $0_R \in S$

2/ $a \in S \Rightarrow -a \in S$

3/ $a, b \in S \Rightarrow a + b \in S$

4/ $a, b \in S \Rightarrow ab \in S$

5/ $1_R \in S$

(b) Define what it means for a ring to be an integral domain.

Solution:

R is an integral domain if

1/ R is non-trivial

2/ R is commutative

3/ $ab = 0_R \Rightarrow a = 0_R$ or $b = 0_R$ $\forall a, b \in R$

(c) Prove that if R is an integral domain then so is any subring $S \subset R$.

Solution:

1/ R non-trivial $\Rightarrow$ $0_R \neq 1_R$. $0_R, 1_R \in S$ $\Rightarrow$ $|S| > 1$ $\Rightarrow$ S non-trivial

2/ R commutative $\Rightarrow$ S commutative

3/ Let $a, b \in S$ and assume $ab = 0_R$

$\Rightarrow$ $a = 0_R$ or $b = 0_R$

$\Rightarrow$ S has no zero divisors

$\Rightarrow$ S integral domain

(d) Give an example of a ring R which is not an integral domain, but contains a subring which is an integral domain.

Solutions:

$R = \mathbb{C}[x]/(x^2)$ $\Rightarrow$ $x + (x^2) \neq 0 + (x^2)$

but $(x + (x^2))(x + (x^2)) = x^2 + (x^2) = 0 + (x^2)$

$\Rightarrow$ R not an integral domain.

$S = \{ \lambda + (x^2) \mid \lambda \in \mathbb{C} \}$

$S \cong \mathbb{C}$. $\mathbb{C}$ is a field so it is an integral domain.

2. (25 points) Let R and S be non-trivial rings.

(a) Define what it means for a map $\phi : R \to S$ to be a ring homomorphism.

Solution:

$\phi : R \to S$ is a ring homomorphism iff

1/ $\phi(a+b) = \phi(a) + \phi(b) \quad \forall a,b \in R$

2/ $\phi(ab) = \phi(a)\phi(b) \quad \forall a,b \in R$

3/ $\phi(1_R) = 1_S$

(b) Prove $Im(\phi) \subset S$ is a subring.

Solution:

$\phi : R \to S$ is a group homomorphism under $+$. Hence $Im\,\phi \subset S$ is a subgroup under $+$.

Let $a,b \in Im\,\phi \Rightarrow \exists r,s \in R$ s.t. $\phi(r) = a$ $\phi(s) = b$

$\Rightarrow ab = \phi(r)\phi(s) = \phi(rs) \in Im\,\phi$

$\phi(1_R) = 1_S \Rightarrow 1_S \in Im\,\phi$

$\Rightarrow Im\,\phi \subset S$ is a subring

(c) Prove that $r \in R^* \Rightarrow \phi(r) \in S^*$. Is the converse true? Be sure to justify your answer.

Solution:

Let $r \in R^* \Rightarrow rr^{-1} = r^{-1}r = 1_R \Rightarrow$

$\phi(r)\phi(r^{-1}) = \phi(r^{-1})\phi(r) = \phi(1_R) = 1_S \Rightarrow \phi(r) \in S^*$

Converse is <u>not</u> true. E.g. $\phi : \mathbb{C}[x] \to \mathbb{C}$, $f(x) \to f(1)$

$\phi(x) = 1 \in \mathbb{C}^*$, however $x \notin (\mathbb{C}[x])^*$

3. (25 points) Let R be an integral domain.

   (a) Define the field of fractions of R, denoted $Frac(R)$. Make sure you define both addition and multiplication. You do not need to prove they are well-defined.

   Solution:

$Frac(R)$ = equivalence classes in $R \times (R \setminus \{0_R\})$ under the relation $(a,b) \sim (c,d) \iff ad - bc = 0_R$

Let $\frac{a}{b} = [(a,b)]$.

$$\frac{a}{b} + \frac{c}{d} := \frac{ad+bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

   (b) Prove that if R is a field then $R \cong Frac(R)$. You may use any result in lectures as long as it is clearly stated.

Fact There is an injective homomorphism $\phi : R \to Frac(R)$, $a \mapsto \frac{a}{1}$

Assume R is a field. Let $\frac{a}{b} \in Frac(R) \Rightarrow b \neq 0_R$

$\Rightarrow b^{-1}$ exists $\Rightarrow \frac{a}{b} = \frac{ab^{-1}}{1} \Rightarrow \phi$ is surjective

$\Rightarrow \phi$ bijective $\Rightarrow R \cong Frac(R)$

4. (25 points) Let R be an integral domain.

(a) Define what it means for $a \in R$ to be irreducible.

Solution:

$a \in R$ is irreducible if

1/ $a \neq 0_R$

2/ $a \notin R^*$

3/ $a = bc \Rightarrow b \in R^*$ or $c \in R^*$

(b) Prove that if $a, b \in R$ are associated then a irreducible $\Rightarrow$ b irreducible.

Solution:

a, b associated $\Rightarrow$ $a = bu$ for some $u \in R^*$

Assume a irreducible and write $b = cd \Rightarrow a = (uc)d$

$\Rightarrow uc \in R^*$ or $d \in R^* \Rightarrow c \in R^*$ or $d \in R^*$

$\Rightarrow b$ irreducible.

(c) prove that $1 + i$ is irreducible in $\mathbb{Z}[i]$. Be sure to justify your answer.

$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \Rightarrow |\alpha| \geq 1 \;\; \forall \alpha \in \mathbb{Z}[i]$

$\Rightarrow \alpha \in (\mathbb{Z}[i])^* \Leftrightarrow |\alpha| = 1 \Leftrightarrow \alpha = \pm 1, \pm i$

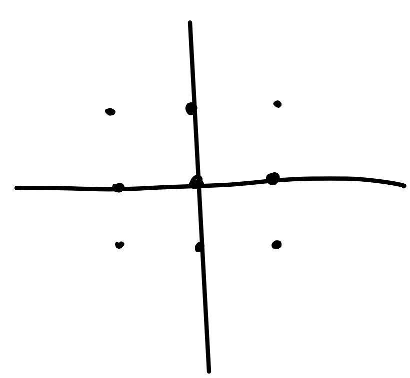
Assume $1 + i = \alpha\beta$ where $\alpha, \beta \in \mathbb{Z}[i] \Rightarrow |\alpha||\beta| = \sqrt{2}$

$\Rightarrow |\alpha|, |\beta| \leq \sqrt{2} \Rightarrow \alpha \in \mathbb{Z}[i], |\alpha| \leq \sqrt{2} \Rightarrow$

$|\alpha| = \sqrt{2}$ or 1. $|\alpha| = 1 \Rightarrow \alpha \in (\mathbb{Z}[i])^*$

$|\alpha| = \sqrt{2} \Rightarrow |\beta| = 1 \Rightarrow \beta \in (\mathbb{Z}[i])^*$

$\Rightarrow 1 + i$ irreducible.



5. (25 points) Prove that a Euclidean ring is a PID.

Solution:

Assume (R, φ) is Euclidean. Let $I \subset R$ be an ideal. $I = \{0_R\} \Rightarrow I = (0_R)$.

Assume $I \neq \{0_R\}$. Choose $b \in I$, $b \neq 0_R$ s.t. $\varphi(b) \in \mathbb{N} \cup \{0\}$ is minimal.

Let $a \in I$. $\Rightarrow a = qb + r$ where $r = 0_R$ ($\Rightarrow a \in (b)$) or $\varphi(r) < \varphi(b)$

$r = a - qb \in I$. Hence $r = 0_R$ by minimality of $\varphi(b)$.

$\Rightarrow I = (b) \Rightarrow R$ is a P.I.D.

6. (25 points) Prove that the quotient ring $\mathbb{Q}[X]/(x^3+x^2+1)$ is a field. You may assume that $x^3+x^2+1 \neq 0$ for all $x \in \mathbb{Q}$, where $|x| > 2$. If you use any results from lectures be sure to state them clearly.

Solution:

$\mathbb{Q}[x]$ a P.I.D. $\Rightarrow$ (x^3+x^2+1) maximal $\Leftrightarrow$ x^3+x^2+1 irreducible.

$\deg(x^3+x^2+1) = 3 \Rightarrow x^3+x^2+1$ reducible $\Leftrightarrow \exists\, \alpha \in \mathbb{Q}$ s.t. $\alpha^3+\alpha^2+1=0$

x^3+x^2+1 monic $\Rightarrow$ $\alpha \in \mathbb{Q}$ a root must be in $\mathbb{Z}$.

$x^3+x^2+1 \neq 0 \quad \forall x \in \mathbb{Q} \quad |x|>2$

$$\left.\begin{array}{l} 2^3+2^2+1 \neq 0 \\ 1^3+1^2+1 \neq 0 \\ 0^3+0^2+1 \neq 0 \\ (-1)^3+(-1)^2+1 \neq 0 \\ (-2)^3+(-2)^2+1 \neq 0 \end{array}\right\}$$

No roots of x^3+x^2+1 in $\mathbb{Z}$

$\Rightarrow$ x^3+x^2+1 irreducible in $\mathbb{Q}[x]$

$\Rightarrow (x^3+x^2+1) \subset \mathbb{Q}[x]$ maximal

$\Rightarrow \mathbb{Q}[x]/(x^3+x^2+1)$ a field.

7. (25 points) (a) Let E/F be a field extension. Define what it means for the extension to be finite.

Solution:

E/F finite means $\dim_F(E) < \infty$. Explicitly this means $\exists\ x_1, \ldots, x_n \in E$ s.t.

$$E = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid \lambda_i \in F\}$$

(b) Prove that E/F finite $\Rightarrow$ E/F algebraic.

Solution:

Let $[E : F] = n$, and $\alpha \in E$.

$\Rightarrow$ $\{1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^n\} \subset E$ must be linearly dependent over F.

$\Rightarrow$ $\exists\ a_0, a_1, \ldots, a_n \in F$, not all zero s.t.

$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0_E$$

$\Rightarrow$ $f(x) = a_0 + a_1 x + \cdots + a_n x^n \neq 0_{F[x]}$ and $f(\alpha) = 0_E$ $\Rightarrow$ α is algebraic over F.

$\Rightarrow$ E/F algebraic.

END OF EXAM