MATH 113 FINAL EXAM (PRACTICE 2)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

This exam consists of 7 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Let R be a ring. Define what it means for a subset $S \subset R$ to be a subring. State all the axioms precisely.

   Solution:

   (b) Define what it means for a ring to be an integral domain.

   Solution:

(c) Prove that if R is an integral domain then so is any subring $S \subset R$.

Solution:

(d) Give an example of a ring R which is not an integral domain, but contains a subring which is an integral domain.

Solutions:

2. (25 points) Let R and S be non-trivial rings.
   (a) Define what it means for a map $\phi : R \to S$ to be a ring homomorphism.
   Solution:

   (b) Prove $Im(\phi) \subset S$ is a subring.
   Solution:

   (c) Prove that $r \in R^* \Rightarrow \phi(r) \in S^*$. Is the converse true? Be sure to justify your answer.
   Solution:

3. (25 points) Let R be an integral domain.

   (a) Define the field of fractions of R, denoted $Frac(R)$. Make sure you define both addition and multiplication. You do not need to prove they are well-defined.

   Solution:

   (b) Prove that if R is a field then $R \cong Frac(R)$. You may use any result in lectures as long as it is clearly stated.

4. (25 points) Let R be an integral domain.

   (a) Define what it means for $a \in R$ to be irreducible.

   Solution:

   (b) Prove that if $a, b \in R$ are associated then a irreducible $\Rightarrow$ b irreducible.

   Solution:

   (c) Prove that $1 + i$ is irreducible in $\mathbb{Z}[i]$. Be sure to justify your answer.

5. (25 points) Prove that a Euclidean ring is a PID.

   Solution:

6. (25 points) Prove that the quotient ring $\mathbb{Q}[X]/(x^3 + x^2 + 1)$ is a field. You may assume that $x^3 + x^2 + 1 \neq 0$ for all $x \in \mathbb{Q}$, where $|x| > 2$. If you use any results from lectures be sure to state them clearly.

   Solution:

7. (25 points) (a) Let E/F be a field extension. Define what it means for the extension to be finite.

   Solution:

   (b) Prove that E/F finite $\Rightarrow$ E/F algebraic.

   Solution:

END OF EXAM