MATH 113 FINAL EXAM (PRACTICE 1)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ____

This exam consists of 7 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Carefully define what it means for a set R to be a ring. State all the axioms precisely.

   Solution:

   (b) Define the units $R^* \subset R$.

   Solution:

(c) Prove, using only the axioms, that $R^* = R$ implies that $|R| = 1$.

Solution:

2. (25 points) Let R be a ring.

   (a) Define what it means for a subset $I \subset R$ to be an ideal.

   Solution:

   (b) Prove that the binary operation

$$\begin{aligned} \phi : R/I \times R/I &\longrightarrow R/I \\ (x+I, y+I) &\longrightarrow (xy)+I \end{aligned}$$

   is well-defined, i.e. independent of coset representative choices.

   Solution:

   (c) If R/I is the quotient ring, is the following true:

   $x + I \in (R/I)^* \Rightarrow x \in R^*$. Be sure to justify your answer.

   Solution:

3. (25 points) Let R be an integral domain.

   (a) Define the characteristic of R.

   Solution:

   (b) Prove that if the characteristic of R is p, then there is an injective homomorphism $\phi : \mathbb{F}_p \to R$. Be sure to carefully justify your answer.

4. (25 points) Let R be a commutative ring.
   (a) Define what it means for two elements $a, b \in R$ to be associated.
   Solution:

   (b) Prove that if R is an integral domain then a and b are associated if and only if there exists $u \in R^*$ such that $a = ub$.
   Solution:

   (c) Using this, prove that $2\sqrt{2}+1$ and $5+3\sqrt{2}$ are associated in $\mathbb{Z}[\sqrt{2}]$.

5. (25 points) Prove that if R is a PID then $a \in R$ is irreducible $\iff$ $(a) \subset R$ is maximal.

Solution:

6. (25 points) Let R be an integral domain.

   (a) Define what it means for an ideal $I \subset R$ to be maximal.

   Solution:

   (b) Is the ideal $(x^4 - 1, x^5 - x^3) \subset \mathbb{Q}[X]$ maximal? Be sure to carefully justify your answer. If you use any results from lecture be sure to state them clearly.

   Solution:

7. (25 points) (a) Let E/F be a field extension and let $\alpha \in E$ be algebraic over F. Define the minimal polynomial of α over F.

   Solution:

   (b) Prove the minimal polynomial is irreducible.

   Solution:

(c) Determine the degree of the extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. You may use any results from lectures as long as they are clearly stated.

END OF EXAM