Math 55 final exam, May 7, 2024

This 180-minute exam included 10 problems, with 60 the maximum possible score:

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Points | 6 | 6 | 7 | 5 | 8 | 6 | 6 | 5 | 6 | 5 | 60 |

The cover sheet had this info:

> It is OK to leave factorials and expressions like $C(n,r)$ or $P(n,r)$ in your answers.
>
> A standard deck of 52 cards has 13 kinds of cards, with four cards of each kind, one in each of the four suits: hearts, diamonds, spades, and clubs.

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This exam featured a number of problems that you'd seen before on homework or practice exams: problem 1 is a rephrasing of a problem from the fall, 1997 final; problem 7 was on the last (optional) homework; problem 8 is related to the "extra" homework problems on probability; problem 9 was on HW 12. Problem 2 was intended to be a straightforward induction problem. Problem 3 was a fairly standard Chinese Remainder Theorem problem. I think that Problem 4 was on a previous exam but didn't spot it when I looked at old exams very quickly.

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**1.** How many strings of length eight can be made by using eight of the ten letters `RRR GGG BBBB`?

**2.** Consider the binomial coefficients $\binom{2n}{n}$ for $n \geq 0$: 1, 2, 6, 20, 70, 252, .... Establish the inequality $\binom{2n}{n} < 2^{2n-2}$ for all $n \geq 5$.

**3a.** Use the equation $1 = 3617 \cdot 4115 - 822 \cdot 18107$ to find an integer $x$ satisfying

$$x \equiv \begin{cases} 123 & \pmod{4115}, \\ 456 & \pmod{18107}. \end{cases}$$

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(You can write $x$ as an arithmetical expression without evaluating the expression.)

**3b.** If $x$ and $x'$ satisfy the two congruences, show that $x \equiv x' \pmod{4115 \cdot 18107}$.

**4.** Oksi tosses a fair coin repeatedly until either two tails in a row have come up (`TT`) or the sequence heads-tails (`HT`) has come up. What is the probability that `HT` (rather than `TT`) is the sequence that stops Oski's tossing?

**5a.** How many five-card poker hands contain no hearts $\heartsuit$ or clubs $\clubsuit$?

**5b.** How many five-card poker hands contain at most two of the four suits?

**6.** Let $L_n$ be the sequence defined by:

$$L_n = \begin{cases} 2 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ L_{n-1} + L_{n-2} & \text{if } n \geq 2. \end{cases}$$

Prove that

$$L_n = f_{n-1} + f_{n+1}$$

for all $n \geq 1$, where $f_n$ is the $n$th Fibonacci number.

**7.** Suppose that $G$ is a simple graph with at least two vertices. Show that there are two vertices of $G$ that have the same degree.

**8.** Alice rolls a fair die repeatedly until the number 6 comes up. What's the probability that 2 comes up on the first roll, given that only even numbers come up as Alice rolls?

**9.** Let $X$ be a random variable on a sample space $S$ such that $X(s) \geq 0$ for all $s \in S$. If $a$ is a positive real number, show that $p(X \geq a) \leq E(X)/a$.

**10.** Consider the set of real numbers between 0 and 1 whose decimal expansions consist of the blocs 00 and 55. For example,

$$.5500550055555555\ldots, \quad .0055, \quad 550055005500550055005500\ldots$$

are three such numbers. Is this set countable? (As for all problems, be sure to explain your reasoning carefully.)

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