Math 54 Professor K. A. Ribet

Final Exam May 18, 2007

This was a 3-hour exam, 12:30–3:30PM. There were 60 points for eight questions. Problem 1 consisted of 11 true–false questions, each worth 1 point. Problems 2–8 had point values 6, 8, 9, 6, 6, 8 and6.

I think that it was a successful exam: people seemed happy, and there weren't problems that were obviously ambiguous. The only misprint that I know about was in the last problem, where "diagonal" should replace "diagonalizable." We announced this in the exam room.

1. *For each statement below, write TRUE or FALSE to the left of the statement. You are not required to justify your reasoning:*

If A is a square invertible matrix, then A and A^{-1} have the same rank.

True: the rank is the size of the matrix A in both cases.

If A is an $m \times n$ matrix and if b is in $\mathbf{R}^m$, there is a unique $x \in \mathbf{R}^n$ for which $\|Ax - b\|$ is smallest.

False: for example, A could be the 0-matrix and b could be 0. Then the length is smallest (namely 0) for all x.

If A is an $n \times n$ matrix, and if v and w in $\mathbf{R}^n$ satisfy $Av = 2v$, $Aw = 3w$, then $v \cdot w = 0$.

False: it's not true in general that eigenvectors for different eigenvalues are perpendicular. We proved this for symmetric matrices, however.

If the dimensions of the null spaces of a matrix and its transpose are equal, then the matrix is square.

True by the rank-nullity theorem, since a matrix and its transpose have the same rank.

If A is a 2×2 matrix, then -1 cannot be an eigenvalue of A^2.

False. For example, if $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then $A^2 = -I_2$.

I liked the linear algebra portion of this course more than the differential equations portion.

This was supposed to be a "free point," but students who gave no answer probably won't get their empty answer marked correct.

If four linearly independent vectors lie in $\text{Span}(\{ w_1, \ldots, w_t \})$*, then* t *must be at least 4.*

Yes, this is true. The dimension of the span of t different vectors is at most t, whereas the dimension of a space containing 4 linearly independent vectors is at least 4.

If B *is invertible, then the column spaces of* A *and* AB *are equal.*

True. The column space of A is the set of all Ax, whereas the column space of AB is the set of all ABy. (Here, x and y are vectors of length n.) Every By is an x. Because B is assumed to be invertible, every x is a By.

If A *is a matrix, the row spaces of* A *and* A^TA *are equal.*

Just as the column space of AB is always contained in the column space of A, so the row space of BA is always contained in the row space of A. In particular, the row space of A^TA is contained in the row space of A. The two spaces are therefore equal if and only if they have the same dimension. You may recall (p. 258 of the linear algebra book) that the null space of A^TA is equal to the null space of A; this follows from a computation with the dot product. A consequence is that A^TA and A have equal ranks. Accordingly, the two row spaces have the same dimension and the assertion is true.

If two symmetric $n \times n$ *matrices* A *and* B *have the same eigenvalues, then* $A = B$*.*

False. For example the diagonal matrix with diagonal entries 1 and 2 has the same eigenvalues as the diagonal matrix with entries 2 and 1 (i.e., in the other order). Both are symmetric; they have the same eigenvalues; they're different.

If the characteristic polynomial of A *is* $(\lambda - 1)(\lambda + 1)(\lambda - 3)^2$*, then* A *is necessarily diagonalizable.*

False because of the repeated eigenvalue.

2. *Consider the vectors* $v_1 = [0,1,0,1,0]$, $v_2 = [0,1,1,0,0]$, $v_3 = [0,1,0,1,1]$ *in* $\mathbf{R}^5$*. Find* w_1, w_2, w_3 *in* $\mathbf{R}^5$ *such that* $w_i \cdot w_j = 0$ *for* $i \neq j$ *(*i *and* j *between 1 and 3), and such that* $\text{Span}(\{ w_1, \cdots, w_i \}) = \text{Span}(\{ v_1, \cdots, v_i \})$ *for* $i = 1, 2, 3$.

You get the ws from the vs by applying a straight Gram–Schmidt operation. Take $w_1 = v_1$. It looks as if w_2 can be $(0, \frac{1}{2}, 1, -\frac{1}{2}, 0)$ and w_3 can be $(0,0,0,0,1)$.

3. *Find* $x_1(t)$ *and* $x_2(t)$ *such that*

$$x_1'(t) = -2x_1(t) + 2x_2(t) \qquad x_2'(t) = +2x_1(t) + x_2(t)$$

and $x_1(0) = -1$, $x_2(0) = 3$.

This is a straightforward $\mathbf{x}' = A\mathbf{x}$ problem. The matrix A is $\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$, whose eigenvalues are 2 and -3. The corresponding eigenvectors are $(1,2)$ and $(2,-1)$. The general solution is

$$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

where C_1 and C_2 are constants. The initial conditions give $C_1 = 1$, $C_2 = -1$.

4. *Let A be the matrix $\begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & 1 & 1 & 0 \\ 4 & 2 & 4 & 2 \end{bmatrix}$. Find bases for each of the following: the null space of A; the row space of A; the column space of A.*

The third row of the matrix is the sum of the first and second rows. This implies that the rank is at most 2. The rank clearly is 2 because the first two rows are not proportional. Thus the null space, row space and column space all have dimension 2. A basis of the row space consists of the first two rows. A basis of the column space is gotten by taking any two columns. A basis of the null space consists of $(1,-4,1,0)$ and $(1,-3,0,1)$.

5. *The theory of Fourier series implies that there are numbers $a_0, a_1, a_2, \ldots$ such that*

$$|\sin x| = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mx$$

for all real numbers x. Find a_0, a_1, a_2 and a_3. (It may be helpful to recall the formula $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$ from trigonometry.)

I got $a_0 = 4/\pi$, $a_1 = 0$, $a_2 = 4\pi/3$, $a_3 = 0$. In fact, $a_n = 0$ for n odd. For n even, a_n is something like $4/\big(\pi(n^2-1)\big)$.

6. *Find $u(x,t)$ that satisfies the equation $25u_{xx} = u_t$ on the region $0 < x < \pi$, $t > 0$ as well as the boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$ and $u(x,0) = \sin 3x - \sin 4x$ for $0 \le x \le \pi$.*

This is a straightforward heat equation problem like the one from my previous exam. The function $f(x) = \sin 3x - \sin 4x$ is already written as a Fourier series. There is absolutely no need to calculate integrals to do this problem. Just write down the answer, which seems to be $e^{-225t} \sin 3x - e^{-400t} \sin 4x$.

7. *Suppose that $v_1, \ldots, v_n$ are vectors in $\mathbf{R}^n$ and that A is an $n \times n$ matrix. If $Av_1, \ldots, Av_n$ form a basis of $\mathbf{R}^n$, show that $v_1, \ldots, v_n$ form a basis of $\mathbf{R}^n$ and that A is invertible.*

If you have n vectors in n-space, they form a basis if and only if they're linearly independent, and they form a basis if and only if they span. You can view the hypothesis as the statement that $Av_1, \ldots, Av_n$ are linearly independent. The proof that $v_1, \ldots, v_n$ are linearly independent was explained in the solutions to MT1. (If you say "We already proved this on the midterm, so I don't have to give the proof here," you will not get credit for the proof.) Since the v_i are linearly independent, they form a basis of $\mathbf{R}^n$. To see the invertibility of A, there are various options. For example, you might want to exploit the theorem (1.49 or something) to the effect that A has an inverse if and only if its null space is 0. Suppose x is in $\mathbf{R}^n$ and $Ax = 0$; we want to prove that $x = 0$. Write $x = c_1v_1 + \cdots + c_nv_n$, which is possible because the v_i span $\mathbf{R}^n$. Then $0 = Ax = c_1Av_1 + \cdots + c_nAv_n$. Because the Av_i are linearly independent, all the c_i are 0. Hence $x = 0$, as required.

8. *Let* $v_1 = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$. *Suppose that* A *is the* 3×3 *matrix for which* $Av_1 = v_1$, $Av_2 = 0$, $Av_3 = 5v_3$. *Find an invertible matrix* S *and a diagonalizable matrix* Λ *such that* $A = S\Lambda S^{-1}$.

The v_i are eigenvectors with eigenvalues 1, 0 and 5. We can take Λ to be the diagonal matrix with diagonal entries 1, 0 and 5. We take S to be the 3×3 matrix whose three columns are v_1, v_2 and v_3, in that order.