Lecture 3

NEWTON–OKOUNKOV BODIES AND ISOPERIMETRIC TYPE INEQUALITIES IN ALGEBRAIC GEOMETRY

Askold Khovanskii
University of Toronto

March 20, 2025

PROBLEM ON THE NUMBER OF SOLUTIONS

A *Laurent polynomial* P with a support $\mathrm{Supp}(P) = A \subset \mathbb{Z}^n$ is a function $P = \sum_{m \in A} c_m x^m$, where $A \subset \mathbb{Z}^n$ is a finite set, $c_m \neq 0$ is a complex number, and $x^m = x_1^{m_1} \dots x_n^{m_n}$.

The *Newton polyhedron* $\Delta(P)$ is the convex hull of $A = \mathrm{Supp}(P)$.

Let L_A be the space generated by x^m, where $m \in A$, i.e., $P \in L_A \Leftrightarrow \mathrm{Supp}\, P \subset A$.

Problem. *How many solutions in $(\mathbb{C}^*)^n$ has a system of equations*

$$P_1 = \dots = P_n = 0,$$

where $P_1, \dots, P_n$ are generic Laurent polynomials with the fixed supports $A_1, \dots, A_n \subset \mathbb{Z}^n$?

THE BKK THEOREM

Theorem (Kouchnirenko). *If all Laurent polynomials P_i have the same support A, i.e.,* $\mathrm{Supp}(P_i) = A_i = A$*, then the number of solutions is equal to $n!V(\Delta)$, where $V(\Delta)$ is the volume of the convex hull Δ of the set A, i.e., $\Delta = \Delta(P_i)$.*

Theorem (Bernstein). *The number of solutions is equal to $n!V_n(\Delta_1, \ldots, \Delta_n)$, where $V_n(\Delta_1, \ldots, \Delta_n)$ is the mixed volume of the polyhedra $\Delta_i = \Delta(P_i)$ (in the other words, of the convex hulls Δ_i of the sets A_i).*

These theorems are often referred to as the BKK (Bernstein, Koushnirenko, Khovanskii) theorem. In the lecture, I will discuss wide generalizations of the BKK theorem and the interplay between algebra and geometry suggested by these generalizations.

SUBGROUPS IN THE LATTICE $\mathbb{R}^n$

Let $\mathbf{b} = \{b_1, \dots, b_r\}$ be a sequence of $r \le n$ natural numbers, such that b_{i+1} is divisible by b_i for each $i < r$.

A *group* $H_{\mathbf{b}} \subset \mathbb{Z}^n$ is the group of points $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$, such that x_i is divisible by b_i for $i \le r$ and $x_i = 0$ for $i > r$.

Theorem. *Any group $H \subset \mathbb{Z}^n$ by an isomorphism of $\mathbb{Z}^n$ can be reduced to a unique subgroup $H_{\mathbf{b}}$. Thus, the sequence $\mathbf{b}$ provides an invariant separating nonequivalent subgroup in $\mathbb{Z}^n$.*

Corollary. *To a commutative group G with n generators one assigns a unique sequence $\mathbf{b}$, such that G is equal to the sum of $n - r$ copies of $\mathbb{Z}$ and the groups $\mathbb{Z}/b_i\mathbb{Z}$ for each member b_i of $\mathbf{b}$.*

To prove Corollary, one represents G as a factor-group $\mathbb{Z}^n/H$ and applies Theorem to the group $H \subset \mathbb{Z}^n$.

FINITELY GENERATED SEMIGROUPS IN $\mathbb{Z}^n$

For a semigroup $S \subset \mathbb{Z}^n$, generated by a finite set $A \subset S$, let:

1) $G(S) \subset \mathbb{Z}^n$ be the group generated by $A \subset S$,

2) $L(S) \subset \mathbb{R}^n$ be the subspace spanned by $A \subset S$,

3) $C(S)$ be the polyhedral convex cone spanned by $A \subset S$.

The inclusion $S \subset C(S) \cap G(S)$ follows from definitions.

Slightly weakened, inclusion in the opposite direction also holds.

Theorem A. *There is a vector $a \in S$ such that the intersection of the shifted cone $(C(S)+a)$ with the group $G(S)$ belongs to S.*

SUMS OF FINITE SET

The sum of subsets A, B in an abelian group G is the set C of points c representable as $c = a + b$, where $a \in A, b \in B$.

For a finite set $A \subset \mathbb{Z}^n$, let $k * A$ be the sum of k copies of A.

Theorem. *If the differences $(x - y)$ of $x, y \in A$ generate $\mathbb{Z}^n$, then*

$$\lim_{k\to\infty} \frac{\#(k * A)}{k^n} = V(\Delta(A)),$$

where $V(\Delta(A))$ is the volume of the convex hull $\Delta(A)$ of A.

Proof uses Theorem A for the semigroup $S_A \subset \mathbb{Z} \times \mathbb{Z}^n$ consisting of points $(k, x) \in \mathbb{Z} \times \mathbb{Z}^n$ such that $x \in k * A$ and the following:

Lemma. *For a convex bounded domain $U \subset \mathbb{R}^n$ we have*

$$\lim_{k\to\infty} \frac{\#(kU \cap \mathbb{Z}^n)}{k^n} = V(U).$$

PROOF OF KOUSHNIRENKO'S THEOREM

Let $A \subset \mathbb{Z}^n$ be a set containing $\#A = N$ characters $\{\chi_i\}$ of $T = (\mathbb{C}^*)^n$ and let $\mathbb{P}^{N-1}$ be a projective space whose homogeneous coordinate are in one-to-one correspondence with the set $\{\chi_i\}$. Let $\Phi_A : T \to \mathbb{P}^{N-1}$ be a map which sends $x \in T$ to

$$\chi_1(x) : \cdots : \chi_N(x) \in \mathbb{P}^{N-1}.$$

Let $Y \subset \mathbb{P}^{N-1}$ be the closure of the image $\Phi_A(T)$ of I in $\mathbb{P}^{N-1}$. The value at m of the *Hilbert function* H_Y of Y is the dimension of space $\Phi_A^*(L^m)$, where L^m is the space of homogeneous polynomials in $\{\chi_I\}$ of degree m. By construction,

$$H_Y(m) = \#(m * A), \quad \text{thus,} \quad \lim_{k\to\infty} \frac{H(k)}{k^n} = V(\Delta(A)),$$

where $V(\Delta(A))$ is the volume of the convex hull $\Delta(A)$ of A.

The Hilbert theorem on degree of variety completes the proof.

SEMIGROUP $K(X)$

Let X be an irreducible n-dimensional variety.

A set $K(X)$ of nonzero finite dimensional spaces of rational functions on X is a multiplicative semigroup with the following multiplication: the product L_1L_2 of $L_1, L_2 \in K(X)$ is the space L_1L_2 generated by functions fg, where $f \in L_1$, $g \in L_2$.

A Zariski open set $U \subset X$ is *admissible* for an n-tuple of spaces $L_1, \dots, L_n \in K(X)$ if the following conditions hold:

a) U does not contain singular points of X;

b) all functions from $L_1, \dots, L_n$ are regular on U;

c) U does not contain *base points* of a space L_i, i.e., for any i, $0 \leq i \leq n$, and any $x \in U$ these is $f \in L_i$ such that $f(x) \neq 0$.

INTERSECTION INDEX ON $K(X)$

For a vector-function $\mathbf{f} = (f_1, \dots, f_n)$, where $f_i \in L_i$, and an admissible set U for $L_1, \dots, Ln$, denote by $\#(\mathbf{f}(x) = 0 | x \in U)$ the number of simple roots $x \in U$ of the system

$$f_1(x) = \dots = f_n(x) = 0.$$

Theorem. *For a given admissible set U and generic vector-function $\mathbf{f}$, the number $\#(\mathbf{f}(x) = 0 | x \in U)$ depends only on n-tuple of spaces $L_1, \dots, L_n$.*

The number from Theorem is called the *intersection index of the spaces $L_1, \dots, L_n$*. We denote it by $[L_1, \dots, L_n]$.

Example. The BKK theorem computes the intersection index $[L_{A_1}, \dots, L_{A_n}]$ on the torus $X = (\mathbb{C}^*)^n$.

PROPERTIES OF THE INTERSECTION INDEX

Let $\mathrm{Deg}(L) = [L, \dots, L]$ be the *self-intersection index* of L.

Theorem. *The following properties hold:*

1. $[L_1, \dots, L_n]$ *is symmetric in permuting the spaces* $L_1, \dots, L_n$*;*

2. $[L_1, \dots, L_n]$ *is linear in each argument. For example,*

$$[L_1' L_1'', L_2, \dots, L_n] = [L_1', L_2, \dots, L_n] + [L_1'', L_2, \dots, L_n].$$

q

Easy to prove the following theorem:

Theorem. *The intersection index is nonnegative, i.e.,*

$[L_1, \dots, L_n] \geq 0,$

and monotone, i.e., if $L_1' \subset L_1, \dots, L_n' \subset L_n$*, then*

$[L_1', \dots, L_n'] \leq [L_1, \dots, L_n].$

REGULARIZATION OF SUB-SEMIGROUPS IN $\mathbb{Z}^n$

For a **semigroup** $S \subset \mathbb{Z}^n$ **of integral points**, let:

1) $G(S) \subset \mathbb{Z}^n$ be the group generated by S;

2) $L(S) \subset \mathbb{R}^n$ be the subspace spanned by S;

3) $C(S)$ be the closure of the convex spanned by S.

The **regularization** $\tilde{S}$ of S is the semigroup $C(S) \cap G(S)$.

Theorem. *Let $C' \subset C(S)$ be a pointed convex cone which intersects the boundary (in the topology of the linear space $L(S)$) of the cone $C(S)$ only at the origin.*

Then there exists a constant $N > 0$ (depending on C') such that any point in the group $G(S)$ which lies in C' and whose distance from the origin is bigger than N belongs to S.

NEWTON–OKOUNKOV BODY (NO BODY)

Let $S \subset \mathbb{Z} \times \mathbb{Z}^n$ be a semigroup such that:

a) $(t, x) \in S \Rightarrow t \geq 0$; b) $C(S) \cap \{t = 0\} = \{0\}$;

c) $S \cap \{t = 1\} \neq \emptyset$.

The **Hilbert function** H_S **of** S is the function on natural numbers defined by an identity $H_S(k) = \#(\{t = k\} \cap S)$.

The Newton–Okounkov body (NO body) of S is the convex body $\Delta(S) = \{t = 1\} \cap C(S)$.

Theorem. *The function $H_S(k)$ grows like $a_q k^q$, where*

$$q = \dim_{\mathbb{R}} \Delta(S)$$

and a_q is equal to the q-dimensional volume of $\Delta(S)$ divide by order of the factor-group $(0, \mathbb{Z}^n)/(0, \mathbb{Z}^n) \cap S$.

$\mathbb{Z}^n$-VALUED VALUATION ON $\mathbb{C}(X) \setminus \{0\}$

Let $\mathbb{Z}^n$ be the lattice equipped with the lexicographic order $>$.

A surjective map $v : \mathbb{C}|(X) \setminus \{0\} \to \mathbb{Z}^n$ is a $\mathbb{Z}^n$-*valuation* if

$$v(fg) = v(f) + v(g) \quad \text{and} \quad v(f+g) \geq \min\{v(f), v(g)\}.$$

Example. Let $a \in X$ be a smooth point and let $x_1, \dots, x_n$ be a system of coordinates about a such that $x_1(a) = \cdots = x_n(a) = 0$.

If f is a regular functions at a, define $v(f)$ as $\mathbf{m} \in \mathbb{Z}^n$, where $\mathbf{x}^{\mathbf{m}}$ is the smallest monomial which appears with nonzero coefficient in the Taylor series of f about the point a.

If f, g are regular at a, then $v(f/g)$ is defined as $v(f) - v(g)$.

NO BODY $\Delta(L)$ OF $L \in K(X)$

Spaces $L_1, L_2 \in K(X)$ are *equivalent* if there is $M \in K(X)$ such that $L_1M = L_2M$. For every $L \in K(X)$, there is the space $\overline{L} \in K(X)$ which is the biggest by inclusion among all spaces equivalent to L.

A $\mathbb{Z}^n$-valuation v assigns to a space $L \in K(X)$ the semigroup $S(L) \subset \mathbb{Z} \times \mathbb{Z}^n$ as follows: (k, x) belongs to $S(L)$ if $x \in v(\overline{L^k})$.

The NO body of $S(L)$ is called the NO body $\Delta(L)$ of L.

Main Theorem. *The following relations hold:*

1. for any $L \in K(X)$, we have:

$$[L, \dots, L] = n!V(\Delta(L));$$

2. for any $L_1, L_2 \in K(X)$, we have:

$$\Delta(L_1L_2) \supset \Delta(L_1) + \Delta(L_2).$$

MIXED VOLUME

On the additive semigroup of convex bodies in $\mathbb{R}^n$*, the volume is a homogeneous polynomial of degree* n*.* It means that if $\Delta_1, \ldots, \Delta_m \subset \mathbb{R}^n$ are convex bodies, then the volume of $\Delta = k_1\Delta_1 + \cdots + k_m\Delta_m$ is a homogeneous polynomial of degree n in m-tuples of natural numbers $(k_1, \ldots, k_m)$.

This property implies the existence and uniqueness of the mixed volume $V_n(\Delta_1, \ldots, \Delta_n)$ on n-tuples of convex bodies such that:

1. on the diagonal it coincides with the volume, i.e., $V(\Delta, \ldots, \Delta)$ is the volume of Δ;
2. V is symmetric in permuting the variables;
3. V is linear in each argument; for example: $V(\Delta_1'+\Delta_1'', \Delta_2, \ldots, \Delta_n) = V(\Delta_1', \Delta_2, \ldots \Delta_n)+V(\Delta_1'', \Delta_2, \ldots \Delta_n)$;

PROPERTIES OF MIXED VOLUME

The following theorem is easy to prove.

Theorem. *The mixed volume V_n is:*

1. *nonnegative, i.e., the mixed volume of any n-tuple $\Delta_1, \dots, \Delta_n$ of convex bodies is nonnegative;*
2. *monotone, i.e., if $\Delta'_1 \subset \Delta_1, \dots, \Delta'_n \subset \Delta_n$, then*

$$V_n(\Delta'_1, \dots, \Delta'_n) \leq V_n(\Delta_1, \dots, \Delta_n).$$

Example. Let $B \subset \mathbb{R}^n$ be a unit ball centered at the origin. Then for any convex body $\Delta \subset \mathbb{R}^n$ the mixed volume

$$V_n(\Delta_1, \dots, \Delta_n),$$

where $\Delta_1 = \cdots = \Delta_{n-1} = \Delta$ and $\Delta_n = B$,

is equal to the $(n-1)$-dimensional volume of the boundary $\partial\Delta$ of Δ multiplied by $\frac{1}{n}$.

CLASSICAL GEOMETRIC INEQUALITIES

Theorem *(Brunn–Minkowski inequality). If Δ_1, Δ_2 are convex bodies in $\mathbb{R}^n$, then*

$$V^{\frac{1}{n}}(\Delta_1) + V^{\frac{1}{n}}(\Delta_2) \leq V^{\frac{1}{n}}(\Delta_1 + \Delta_2). \tag{1}$$

The Brunn–Minkowski inequality has visual geometric proofs.

Theorem (Alexandrov–Fenchel inequality). *For any n-tuple of convex bodies $\Delta_1, \Delta_2, \dots, \Delta_n$ in $\mathbb{L}_n$ the following inequality holds:*

$$\begin{aligned} V_n^2(\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_n) \leq \\ V_n(\Delta_1, \Delta_1, \Delta_3, \dots, \Delta_n) V_n(\Delta_1, \Delta_1, \Delta_3, \dots, \Delta_n). \end{aligned} \tag{2}$$

The Alexandrov–Fenchel inequality is one of the most general inequalities between mixed volumes of convex bodies. It has many corollaries. Below are two examples.

COROLLARIES

Corollary 1. *For any* $2 \leq m \leq n$ *and for any* n*-tuple of convex bodies* $\Delta_1, \dots, \Delta_n$*, we have:*

$$\prod_{1 \leq i \leq m} V_n(\Delta_i, \dots, \Delta_i, \Delta_{m+1}, \dots, \Delta_n) \leq V_n(\Delta_1, \dots, \Delta_n)^m. \tag{3}$$

For $m = 2$, inequality (3) coincides with the inequality (2).

Corollary 2. *For any* $2 \leq m \leq n$ *and for any collection of convex bodies* $\Delta_1, \Delta_2, \Delta_{m+1}, \dots, \Delta_n$ *we have:*

$$\begin{aligned} V_n^{\frac{1}{m}}(\Delta_1, \dots, \Delta_1, \Delta_{m+1}, \dots, \Delta_n) + \\ V_n^{\frac{1}{m}}(\Delta_2, \dots, \Delta_2, \Delta_{m+1}, \dots, \Delta_n) \leq \\ V_n^{\frac{1}{m}}(\Delta_1 + \Delta_2, \dots, \Delta_1 + \Delta_2, \Delta_{m+1}, \dots, \Delta_n). \end{aligned} \tag{4}$$

For $m = 2$, inequality (4) coincides with the inequality (3).

INEQUALITIES IN DIMENSION TWO

Let $Q : S \to \mathbb{R}$ be a homogeneous polynomial of degree two, let $B : S^2 \to \mathbb{R}$ be its polarization.

Then for a *pair of points* $x, y \in S$ *the polynomial* Q *satisfies*:

1. *the Brunn–Minkowski type inequality* if $Q(x) \geq 0$, $Q(y) \geq 0$, $Q(x+y) \geq 0$ and

$$Q^{\frac{1}{2}}(x) + Q^{\frac{1}{2}}(y) \leq Q^{\frac{1}{2}}(x+y); \tag{5}$$

2. *the Alexandrov–Fenchel type inequality* if $Q(x) \geq 0$, $Q(y) \geq 0$, $B(x,y) \geq 0$ and

$$Q(x)Q(y) \leq B^2(x,y). \tag{6}$$

EQUIVALENCE OF THE TWO INEQUALITIES

The following theorem is easy to prove:

Theorem. *A homogeneous polynomial $Q : S \to \mathbb{R}$ of degree 2 satisfies the Brunn–Minkowski type inequality for x, y if and only if it satisfies the Alexandrov–Fenchel type inequality for x, y.*

Proof. Assume that Q satisfies the Brunn–Minkowski type inequality for $x, y \in S$, squaring both sides of (5), we obtain $Q(x) + 2Q(x)^{\frac{1}{2}}Q(y)^{\frac{1}{2}} + Q(y) \leq Q(x) + 2B(x, y) + Q(y)$, or $\quad Q(x)^{\frac{1}{2}}Q(y)^{\frac{1}{2}} \leq B(x, y)$.

Thus, $B(x, y)$ is nonnegative. By squaring both sides of the previous inequality, we obtain $Q(x)Q(y) \leq B^2(x, y)$.
Theorem is proven in one direction. Its proof in the opposite direction is similar.

ISOPERIMETRIC INEQUALITY

Our proofs of inequalities in algebra and geometry are based on the Brunn–Minkowski inequality for $n = 2$. To prove it, one either can refer to the Brunn–Minkowski inequality for any n (which is not hard to prove) or to the Alexandrov–Fenchel inequality for $n = 2$, which is as easy to prove as the isoperimetric inequality.

Corollary 1. *The area $V(\Delta)$ of a convex body $\Delta \subset \mathbb{R}^2$ and the length $l(\partial\Delta)$ of its boundary satisfy the following inequality:*

$$V(\Delta) \leq \frac{1}{4\pi} l(\partial\Delta)^2.$$

Moreover, if Δ is a ball, then the inequality becomes equality.

Proof. Corollary follows from Example and from inequality (2) for $n = 2$; $\Delta_1 = \Delta$; $\Delta_2 = B_1$, where B_1 is the unit ball. □

BRUNN–MINKOWSKI TYPE INEQUALITY

Theorem. *Let X be an irreducible n-dimensional variety, L_1, L_2 be elements in $K(X)$ and let L_3 be their product, i.e., $L_3 = L_1 L_2$. Then the following inequality holds:*

$$[L_1, \dots, L_1]^{\frac{1}{n}} + [L_2, \dots, L_2]^{\frac{1}{n}} \leq [L_3, \dots, L_3]^{\frac{1}{n}}.$$

Proof. Let $\Delta(L_1)$, $\Delta(L_2)$, $\Delta(L_3)$ be the NO bodies L_1, L_2, L_3. By Main Theorem, we have the following inclusion:

$$\Delta(L_1) + \Delta(L_2) \subset \Delta(L_3).$$

By the Brunn–Minkovsky inequality, we have:

$$V^{\frac{1}{n}}(\Delta(L_1)) + V^{\frac{1}{n}}(\Delta(L_2)) \leq V^{\frac{1}{n}}(\Delta(L_3)).$$

By Main Theorem for $i = 1, 2, 3$, we have:

$$[L_i, \dots L_i] = n! V(\Delta(L_i)).$$

These relations imply the theorem. □

HODGE TYPE INEQUALITY

Theorem. *Let X be an irreducible algebraic surface, L_1, L_2 be elements in $K(X)$ and let L_3 be their product, i.e., $L_3 = L_1 L_2$. Then the following inequality holds:*

$$[L_1, L_1][L_2, L_2] \leq [L_1, L_2]^2.$$

Proof. By Brunn–Minkowski type inequality, we have

$$[L_1, L_1]^{\frac{1}{2}} + [L_2, L_2]^{\frac{1}{2}} \leq [L_1 L_2]^{\frac{1}{2}}.$$

As we proved, this inequality is quadrivalent to the inequality:

$$[L_1, L_1][L_2, L_2] \leq [L_1, L_2]^2.$$

□

Note that for reducible surfaces the Hodge type inequality in general does not hold.

KODAIRA MAP

A point $x \in X$ which is regular for functions f from $L \in K(X)$ defines a linear function l_x on L whose value at $f \in L$ is $f(x)$. One defines a rational map $l : X \to L^*$ where $l(x) = l_x$. The Kodaira map $\Phi_L : X \to \mathbb{P}(L^*)$ is the projectivization of l.

$L \in K(X)$ is a *big space* if the Kodaira map provides a birational isomorphism between X and the closure $Y \subset \mathbb{P}(l^*)$ of $\Phi_L(X)$.

Lemma. *Assume that* X *is an irreducible variety with* $\dim_{\mathbb{C}} X > 1$ *and* $L \in K(X)$ *is a big space. Then a generic hyperplane section of the projective variety* $Y \subset \mathbb{P}(L^*)$, $Y = \overline{\Phi_L(X)}$ *is an irreducible variety.*

Proof. Lemma is a version of the classical Bertini–Lefschetz theorem.

ALEXANDROV–FENCHEL TYPE INEQUALITY

Theorem. *Let X be an irreducible n-dimensional variety, and let $L_1, \dots, L_n$ be elements in $K(X)$. Then we have:*

$$[L_1, L_2, L_3, \dots, L_n]^2 \geq [L_1, L_1, L_3, \dots, L_n][L_2, L_2, L_3, \dots, L_n].$$

Proof. Assume first that $L_3, \dots, L_n$ are very big. spaces. Let U be an admissible set for $L_1, \dots, L_n$ and let $\mathbf{f} = (f_3, \dots, f_n)$ be a generic vector function such that $f_3 \in L_3, \dots, f_n \in L_n$. Then the variety $X_{\mathbf{f}}$ defined in U by $f_3 = \cdots = f_n = 0$ is a smooth surface.

Since the spaces $L_3, \dots, L_n$ are very big by the Bertini–Lefschetz theorem the surface $X_{\mathbf{f}}$ is irreducible.

Theorem follows from the Hodge type inequality for the restriction of spaces L_1, L_2 to the irreducible surface $X_{\mathbf{f}}$.

CONTINUATION

In the proof of the Alexandrov–Fenchel type inequality one can drop the assumption that the spaces $L_3, \dots, L_n$ are very big.

Indeed, let $L_1, \dots, L_n \in K(X)$ be arbitrary elements. Then for any very big space $L \in K(X)$ and for any natural number q,

$$L_1^q L, \dots, L_n^q L$$

are very big spaces Thus for them the Alexandrov–Fenchel type inequalities hold.

The intersection index $[L_1^q L, \dots, L_n^q L]$ is a polynomial in q with the leading term

$$q^n [L_1, \dots, L_n].$$

This argument proves that the Alexandrov–Fenchel type inequalities for n-tuples of spaces $L_1^q L, \dots, L_n^q L$ imply the Alexandrov–Fenchel type inequality for the n-tuple $L_1, \dots, L_n$.

RELATED INEQUALITIES

Let X be an irreducible n-dimensional variety.

Corollary 1. *For any* $2 \leq m \leq n$ *and for any* n*-tuple of spaces* $L_1, \dots, L_n \in K(X)$ *we have:*

$$\prod_{1 \leq i \leq m} [L_i, \dots, L_i, L_{m+1}, \dots, L_n] \leq [L_1, \dots, l_n]^m.$$

For $m = 2$, Corollary 1 gives the Alexandrov–Fenchel type inequality. For $m = n$ it is symmetric in its arguments.

Corollary 2. *For any* $2 \leq m \leq n$ *and for any collection of paces* $L_1, L_2, L_{m+1}, \dots, L_n \in K(X)$ *we have:*

$$[L_1, \dots, L_1, L_{m+1}, \dots, L_n]^{\frac{1}{m}} + [L_2, \dots, L_2, L_{m+1}, \dots, L_n]^{\frac{1}{m}} \leq [L_1 L_2, \dots, L_1 L_2, L_{m+1}, \dots, L_n]^{\frac{1}{m}}.$$

For $m = 2$, Corollary 2 gives the Brunn–Minkowski type inequality.

ALGEBRAIC RESULTS IMPLY GEOMETRIC ONES

The Alexandrov–Fenchel inequality for integral convex polyhedra follows from the Alexandrov–Fenchel type inequality in algebra via the BKK theorem. It implies such inequality for rational polyhedra by homogeneity. Any convex body can be approximated by rational polyhedra. It proves the Alexandrov–Fenchel inequality for any n-tuple of convex bodies.

INTERSECTION INDEX OF DIVISORS

The above inequalities hold for the intersection index of nef Cartier divisors on irreducible projective varieties . They also hold for the birationally invariant intersection index of nef type Shokurov (b)-divisors on irreducible algebraic varieties.

LITERATURE

A detailed presentation of the theory of Newton–Okounkov bodies can be found in:

[1] Kaveh, K. and Khovanskii, A. *Newton–Okounkov bodies, semigroups of integral points, graded algebras and intersection theory.* Ann. of Math. 176 (2012), no. 2, 925–978.

For an overview of the geometric type inequalities in algebraic geometry see:

[2] Khovanskii, A. *Semigroups, Cartier divisors and convex bodies.* arXiv:2502.13099[math.AG].