Lecrute 2

NEWTON POLYHEDRA AND VECTOR-VALUED LAURENT POLYNOMIALS

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NOTATIONS

T is the n-dimensional complex torus $(\mathbb{C}^*)^n$

$$\mathbf{x} = (x_1, \ldots, x_n) \in T \Leftrightarrow x_1 \neq 0, \ldots, x_n \neq 0.$$

$$\mathbf{x}^\alpha = x_1^{a_1} \cdots x_n^{a_n}$$

is a **character** (monomial) **of the power**

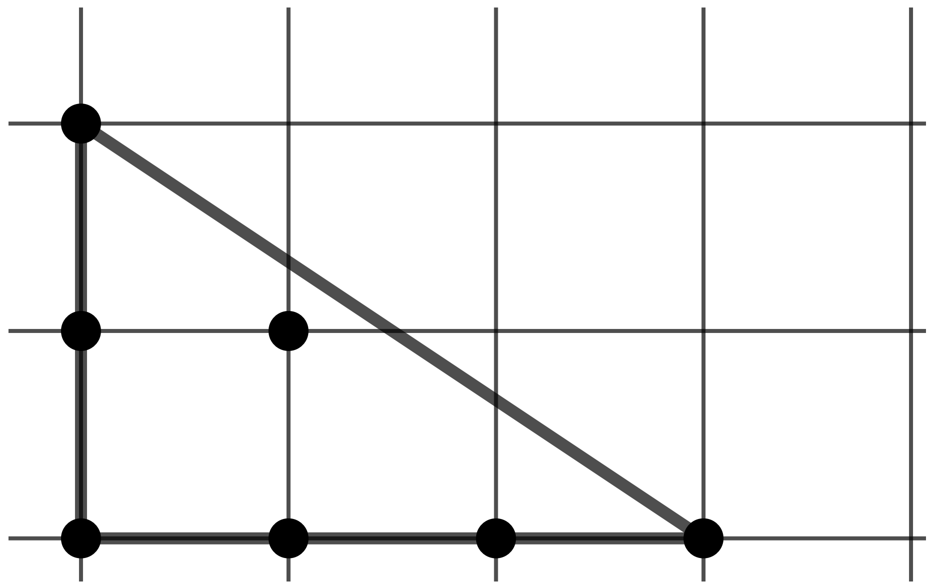
$$\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^n.$$

The *support*, Supp(P), of a **Laurent polynomial** P is the finite set $A \in \mathbb{Z}^n$, such that $P = \sum_{\alpha \in A} c_\alpha \mathbf{x}^\alpha$, where $c_\alpha \neq 0$.

The **Newton polyhedron** $\Delta(P)$ is the convex hull of Supp(P).

EXAMPLE

Let P be $y^2 + a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where $a_0 \neq 0$, $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$. Then $\Delta(P)$ is



and $\mathrm{Supp}(P) = \{(0,0),(0,1),(0,2),(0,3),(2,0)\}$.

FINITE DIMENSIONAL T-INVARIANT SPACES

Let $\mathbb{C}[T]$ be the ring of regular functions on the torus T, i.e., let $\mathbb{C}[T]$ be the ring of Laurent polynomials in $x \in T$.

For a finite set $\mathcal{A}$ in the lattice $\mathbb{Z}^n$, let $L_{\mathcal{A}}$ be the space spanned by characters x^α for $\alpha \in \mathcal{A}$.

Thus a Laurent polynomial P belongs to the space $L_{\mathcal{A}}$ if and only if the inclusion $\mathrm{Supp}(P) \subset \mathcal{A}$ holds.

The torus T acts naturally on the ring $\mathbb{C}(T)$. Each space $L_{\mathcal{A}}$ is a Finite dimensional T-invariant subspaces of the ring $\mathbb{C}(T)$.

Vice versa, any T-invariant finite dimensional subspace of $\mathbb{C}[T]$ is the space $L_{\mathcal{A}}$ for some finite $\mathcal{A} \subset \mathbb{Z}^n$.

MAIN PROBLEM OF THE THEORY

Problem 1. *For generic* $P_1 \in L_{\mathcal{A}_1}, \ldots, P_r \in L_{\mathcal{A}_r}$ *determine the discrete invariants of* $X \subset T$ *defined by the system*

$$P_1 = \cdots = P_r = 0.$$

It turns out that the invariants from Problem 1 depend not on supports $\mathcal{A}_i$ of the Laurent polynomials P_i but only on their Newton polyhedra $\Delta_i = \Delta(P_i)$. Thus Problem 1 is equivalent to the following

Problem $1'$**.** *For generic* $P_1, \ldots, P_r$ *with Newton polyhedra* $\Delta_1, \ldots, \Delta_r$ *determine the discrete invariants of* $X \subset T$ *defined by*

$$P_1 = \cdots = P_r = o.$$

CURVE X DEFINED IN $(\mathbb{C}^*)^2$ BY $P = 0$ WITH $\Delta(P) = \Delta$

Theorem. *For generic P, let $\overline{X} = X \cup X_\infty$ be a smooth compact curve, where X_∞ is a finite set. Then:*

1) the **genus** $g(\overline{X})$ is equal to the number $B(\Delta)$ of integral points in the interior of $\Delta = \Delta(P)$;

2) the number $\# X_\infty$ is equal to the number of integral points in the boundary of Δ;

3) the **Euler characteristic** $\chi(X)$ of X is equal to the volume $V(\Delta)$ of Δ multiplied by $-2!$.

TOY GEOMETRIC APPLICATION

The following formula from planner geometry is well-known:

The Pick formula. *The area of a planar integral polygon* Δ *is equal to the number of integral points inside* Δ *plus half of the number of integral points in its boundary* $\partial\Delta$ *minus one.*

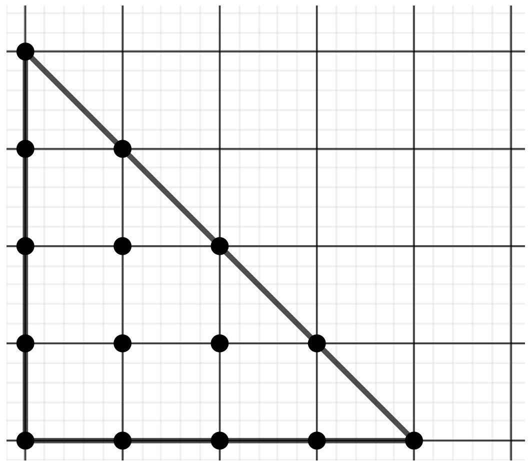
The invariants 1)-3) of an algebraic curve X are related:

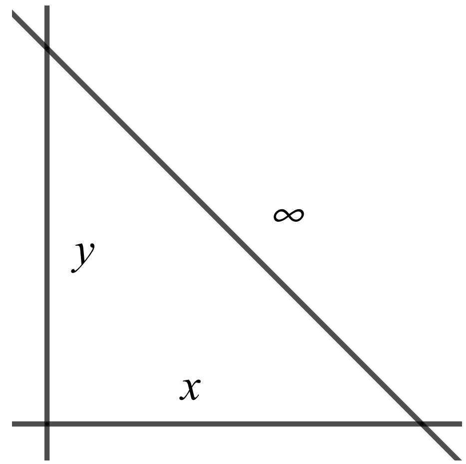
$$\chi(\overline{X}) = \chi(X) + \#(X_\infty) = 2 - 2g(\overline{X}).$$

Thus Theorem from the previous slide implies the **Pick formula** for a planar integral polyhedron Δ:

$$V(\Delta) = \#\left((\Delta \setminus \partial\Delta) \bigcap \mathbb{Z}^2\right) + \frac{1}{2}\#\left(\partial\Delta \bigcap \mathbb{Z}^2\right) - 1.$$

PROJECTIVE PLANE AND POLYNOMIALS





On the left: $\Delta(P)$ for generic polynomial P of degree 5.
On the right: diagram of projective plane.

Terms on the horizontal side of $\Delta(P)$ determine $P(x, 0)$.

Terms on the vertical side of $\Delta(P)$ determine $P(o, y)$.

Terms on the hypotenuse of $\Delta(P)$ determine asymptotic of P about the infinite line.

TORIC VARIETIES

Toric variety M of dimension n is a normal irreducible variety equipped with an action of $(\mathbb{C}^*)^n$ with one orbit $O \sim (\mathbb{C}^*)^n$ Under this action M is broken up into a finite number of orbits isomorphic to tori of different dimensions.

Fan $\Delta^\perp \subset (\mathbb{R}^n)^*$ **dual to** $\Delta \subset \mathbb{R}^n$ is a decomposition of $(\mathbb{R}^n)^*$ into polyhedral cones $\Gamma^\perp$ dual to faces Γ of Δ. A covector ξ lies in the interior of $\Gamma^\perp$ if the function $\langle \xi, x \rangle$ attains minima on Δ at Γ.

To $\Delta^\perp$ **one associates a projective toric variety** M_Δ. Each face $\Gamma \subset \Delta$ corresponds to an orbit $O_\Gamma \subset M_\Delta$ such that:

1) $\dim_{\mathbb{C}} O_\Gamma = \dim_{\mathbb{R}} \Gamma$;

2) if $\Gamma_1 \subset \Gamma_2$, then O_{Γ_1} belongs to the closure of O_{Γ_2}.

Subvariety X **IN** $(\mathbb{C}^*)^2$ **DEFINED BY** $P_1 = \cdots = P_r = 0$ **with** $\Delta(P_1) = \Delta_1, \ldots, \Delta(P_r) = \Delta_r$.

Let $M_{\mathcal{F}} \supset (\mathbb{C}^*)^n$ be a smooth toric variety whose fan $\mathcal{F}$ is a subdivision of the dual fan $\Delta^{\perp}$ to $\Delta = \Delta_1 + \cdots + \Delta_r$.

Theorem A. *The closure of* X *in* $M_{\mathcal{F}}$ *is a smooth variety which is transversal to all orbits of the toric variety* $M_{\mathcal{F}}$.

Theorem A helps to determine many invariants of X. Thus, with V. Danilov we discovered an algorithm for computing Hodge numbers for the mixed Hodge structure on a cohomology ring of X.

Eventually, this algorithm helped to obtain closed formulae for these numbers. V. Batyrev made use of these formulas in his works on mirror symmetry.

THE BKK THEOREM

Theorem (Kouchnirenko). *For generic $P_1, \ldots, P_n$ with the same Newton polyhedron Δ, the system*

$$P_1(x) = \cdots = P_n(x) = 0, \quad x \in (\mathbb{C}^*)^n$$

has simple roots only and their number is $n!V(\Delta)$, where V is the standard volume in $\mathbb{R}^n$.

Theorem (D. Bernstein). *For generic $P_1, \ldots, P_n$ with Newton polyhedra $\Delta_1, \ldots, \Delta_n$, the system*

$$P_1(x) = \cdots = P_n(x) = 0, \quad x \in (\mathbb{C}^*)^n$$

has simple roots only and their number is $n!V_n(\Delta_1, \ldots, \Delta_n)$, where V_n is the Minkowski mixed volume in $\mathbb{R}^n$.

These theories are often referred to as Bernstein–Koushnirenko-Khovanskii (BKK) theorem. In the last 50 year, many generalizations of the BKK theorem were discovered.

THE MINKOWSKI MIXED VOLUME

$(\exists\,!)$ $V_n(\Delta_1,\dots,\Delta_n)$, on n-tuples of convex bodies in $\Delta_i \subset \mathbb{R}^n$, such that:

1. $V_n(\Delta,\dots,\Delta)$ is the volume of Δ;
2. V_n is symmetric;
3. V_n is multi-linear, for example: $V_n(\Delta_1' + \Delta_1'', \Delta_2, \dots) = V_n(\Delta_1', \Delta_2, \dots) + V_n(\Delta_1'', \Delta_2, \dots)$;
4. $0 \le V_n(\Delta_1,\dots,\Delta_n)$;
5. $\Delta_1' \subseteq \Delta_1, \dots, \Delta_n' \subseteq \Delta_n \Rightarrow V_n(\Delta_1',\dots,\Delta_n') \le V_n(\Delta_1,\dots,\Delta_n)$;
6. Alexandrov–Fenchel inequality $V_n^2(\Delta_1,\Delta_2,\dots,\Delta_n) \ge V_n(\Delta_1,\Delta_1,\dots,\Delta_n)V_n(\Delta_2,\Delta_2,\dots,\Delta_n)$;
7. isoperimetric inequality ($n = 2$, Δ_1 is the unite ball, $\Delta = \Delta_2$)
$$\left(\frac{1}{2}\text{ length of }\partial\Delta\right)^2 \ge \pi V(\Delta).$$

VECTOR-VALUED VERSION OF THE THEORY

A **vector-valued Laurent polynomial**, whose values lie in a finite dimensional complex vector space $\mathcal{E}$, is an element of the space

$$\mathcal{E}_T = \mathcal{E} \otimes \mathbb{C}[T].$$

The torus T acts on $\mathcal{E}_T$ by acting on $\mathbb{C}[T]$. For $\alpha \in \mathbb{Z}^n$, let E_α be a subspace in $\mathcal{E}$ which is nonzero only for finitely many α.

The *set* $\{E_\alpha\}$*,* $\alpha \in \mathbb{Z}^n$*, defines a subspace*

$$\mathbf{L} = \oplus_\alpha E_\alpha \otimes x^\alpha$$

of $\mathcal{E}_T$ which is T-invariant and finite dimensional.

Conversely, *any finite dimensional* T*-invariant* $\mathbf{L} \subset \mathcal{E}_T$ *can be defined by some set* $\{E_\alpha\}$.

MAIN PROBLEM OF THE THEORY

The *rank of* $\mathbf{L}$ is the dimension of the space $E = \oplus_\alpha \{E_\alpha\}$.

For $\mathbf{f} \in \mathbf{L}$, let $Y(\mathbf{f}) \subset T$ be the variety defined by the vector equation $\mathbf{f}(x) = 0$.

Problem 2. For generic $\mathbf{f} \in$ Ł, determine the discrete invariants of the variety $Y(\mathbf{f})$ in terms of the discrete invariants of the T-invariant space $\mathbf{L}$.

If $\{E_\alpha\}$ contains only coordinate subspaces of E (with respect to some bases), then Problem 2 is equivalent to Problem 1. Otherwise, it provides its wide generalization.

We'll start with discrete invariants of $\mathbf{L}$.

CHARACTERISTIC SEQUENCE OF L

Let $\mathbf{L}$ be a space of the rank r defined by the set $\{E_\alpha\}$.

An i-tuple $(\alpha_1, \ldots, \alpha_i)$ of integral points is **admissible for L** if one can pick

$$e_j \in E_{\alpha_j} \quad 1 \leq j \leq \ i,$$

such that $\{e_1, \ldots, e_i\}$ is linearly independent.

For $1 \leq i \leq r$, the **characteristic polyhedron** Δ_i is the convex hull of all points α representable as the sum $\alpha_1 + \cdots + \alpha_i$ of elements of an admissible i-tuple for $\mathbf{L}$.

The sequence $\Delta_1, \ldots, \Delta_r$ of characteristic polyhedra form the **characteristic sequence of L.** Its role for Problem 2 is similar to the role of the set of Newton polyhedra for Problem $1'$.

SUPPORT FUNCTION OF L

Let $\mathbf{L}$ be the space defined by a set $\{E_\alpha\}$. For $\xi \in (\mathbb{R}^n)^*$ and $c \in \mathbb{R}$, let $E_{\xi,c}$ be the space defined by the following identity:

$$\oplus_{\langle \xi,\alpha\rangle \leq c} E_\alpha = E_{\xi,c}.$$

For fixed $\xi \in (\mathbb{R}^n)^*$, the **spaces** $E_{\xi,c}$ **form a filtration in** E. A number c_0 is **critical of multiplicity** $k(c_0)$ **for** ξ if

$$\dim_{\mathbb{C}} E_{\xi,c_0} / \oplus_{c<c_0} E_{\xi,c_0} = k(c_0) > 0.$$

For each $\xi \neq 0$**, we have** $\sum_{c_0 \in \mathbb{R}} k(c_0) = r = \dim_{\mathbb{C}} E$.

A **value of** r**-valued support function** $\mathbf{h_L}$ of $\mathbf{L}$ at ξ is the multiset of critical numbers c_0 of ξ, where c_0 are repeated $k(c_0)$ times.

REPRESENTATION OF SUPPORT FUNCTION

A sequence $h_1, \dots, h_r$ of piecewise linear functions *represents* an r-valued support function $\mathbf{h}_{\mathbf{L}}$ if for each ξ the multisets

$$\{h_i(\xi)\} \quad \text{and} \quad \mathbf{h}(\xi)$$

coincide. A sequence $h_1, \dots, h_r$, representing $\mathbf{h}_{\mathbf{L}}$, is not unique, but many invariants of such sequences are equal.

Example. If the rank r of $\mathbf{L}$ is equal to n, then one can define **mixed volume** $V_n(\mathbf{h})$ of the space $\mathbf{L}$ as the mixed volume of a sequence of virtual polyhedra whose support functions $h_1, \dots, h_n$ represent $\mathbf{h}_{\mathbf{L}}$.

One can show that the mixed volume of $\mathbf{h}_{\mathbf{L}}$ is well-defined, i.e., is independent of a choice of the sequence $\{h_1, \dots, h_n\}$ which represents $\mathbf{h}_{\mathbf{L}}$.

DISCRETE INVARIANTS OF A SPACE L

The characteristic sequence $\Delta_1, \ldots, \Delta_r$ and the support function $\mathbf{h_L}$ of $\mathbf{L}$ carry the same discrete information about the space $\mathbf{L}$.

The support function $\mathbf{h_L}$ of a space $\mathbf{L}$ of rank r has a unique representation as a sequence $h_1(\mathbf{L}), \ldots, h_r(\mathbf{L})$ of increasing piecewise linear functions, i.e., $h_1(\mathbf{L}) \leq \cdots \leq h_r(\mathbf{L})$.

Theorem. *The increasing sequence $h_1(\mathbf{L}), \ldots, h_r(\mathbf{L})$ of piecewise linear functions representing the support function $\mathbf{h_L}$ of a space $\mathbf{L}$ of rank r can be defined as follows:*

$$h_i(\mathbf{L}) = h_{\Delta_i} - h_{\Delta_{i-1}},$$

where h_{Δ_i} for $1 \leq i \leq r$ is the support function of the characteristic polyhedron Δ_i, and $h_{\Delta_0} = 0$.

TORIC VARIETY $M_{\mathcal{F}}$ CONVENIENT FOR L

A toric variety $M_{\mathcal{F}} \supset T$ is **convenient** for $\mathbf{L}$ if its fan $\mathcal{F}$ is a subdivision of the normal fan $\Delta^{\perp}$ of

$$\Delta = \Delta_1 + \cdots + \Delta_r,$$

where $\Delta_1, \ldots, \Delta_r$ is the characteristic sequence for the space $\mathbf{L}$.

Theorem B. *If a toric variety $M_{\mathcal{F}}$ is smooth and convenient for the space $\mathbf{L}$, then for generic vector-function $\mathbf{f} \in \mathbf{L}$, the closure in $M_{\mathcal{F}}$ of the variety $Y(\mathbf{f}) \subset T$ is smooth and transversal to all orbits of $M_{\mathcal{F}}$.*

Theorem B relates vector-valued Laurent polynomials with the theory of toric varieties. It has a version which can be stated in terms of T-equivariant vector bundles over toric varieties.

T-EQUIVARIANT TORIC VECTOR BUNDLES

Theorem C. *If $M_{\mathcal{F}}$ is convenient for $\mathbf{L}$, then there is a unique T-equivariant vector bundle $\mathcal{E}$ over $M_{\mathcal{F}}$, such that:*

1) any $\mathbf{f} \in \mathbf{L}$ corresponds to a section of $\mathcal{E}$ over T which can be extended to the global regular section $s(\mathbf{f})$ of $\mathcal{E}$;

2) for any $x \in M_{\mathcal{F}}$ and any vector v in the fiber over x, there is $\mathbf{f} \in \mathbf{L}$ such that $v = s(\mathbf{f})(x)$;

3) the multi-valued function associated to the T-invariant vector bundle $\mathcal{E}$ in the Klyachko's classification is equal to $\mathbf{h}_{\mathbf{L}}$.

Corollary. *The equivariant Chern roots of the vector bundle $\mathcal{E}$ over $M_{\mathcal{F}}$ from the previous theorem are represented by the functions $\{h_1, \ldots, h_r\}$.*

GENERALIZATION OF THE BKK THEOREM

Theorem (generalized BKK theorem). *Assume that the space* $\mathbf{L}$ *has rank* $r = n$. *Then, for generic vector-function* $\mathbf{f} \in \mathbf{L}$, *all points in* $Y(\mathbf{f})$ *are non-degenerate and their number is equal to* $n! MV_n(\mathbf{h_L})$.

In particular, it is equal to

$$n! MV_n(\Delta_1, \Delta_2 - \Delta_1, \ldots, \Delta_n - \Delta_{n-1}).$$

Corollary. Assume that $\mathbf{L}$ has rank $r \leq n$. Then, for generic $\mathbf{f} \in \mathbf{L}$, one can obtain an explicit formula for the class of the variety $Y(\mathbf{f})$ in the ring of conditions of the torus T in terms of the support function $\mathbf{h_L}$.

HYPERPLANES ARRANGEMENT

Let $\{H_1, \dots, H_N\}$ be a hyperplane arrangement in the projective space $\mathbb{P}^n$, where $N > n$. Each hyperplane H_i is defined by $u_i = 0$, where u_i is a linear function in the homogeneous coordinates in $\mathbb{P}^n$.

A Lauret polynomial P in $u_1, \dots, u_N$ whose monomials have degree zero is a rational function on $\mathbb{P}^n$ which is regular on $\mathbb{P}^n \setminus \cup H_i$.

Problem 3. How many solutions has a generic system

$$P_1 = \cdots = P_n = 0$$

in $\mathbb{P}^n \setminus \cup H_i$, where P_i are Laurent polynomials in $u_1, \dots, u_N$, whose Newton polyhedra $\tilde{\Delta}_1, \dots, \tilde{\Delta}_n \subset \mathbb{R}^N$ lye in thee hyperplane

$$x_1 + \cdots + x_N = 0.$$

CHARACTERISTIC SEQUENCE FOR $\{H_1, \dots, H_N\}$

A subsets $J \subset \{1, \dots, N\}$ is *i-admissible* for some i, such that $1 \leq i < N-n$ if it contains i elements and the following condition holds:

$$\cap_{j \notin J} H_j = \emptyset.$$

For an i-admissible set J, let $A_J \in \mathbb{R}^N$ be the point with the coordinate $x_j(A_J) = 1$ if $j \in J$, and $x_j(A_J) = 0$ if $j \notin J$.

The *characteristic sequence for $H_1, \dots, H_N$ is the sequence of polyhedra $\Delta_1, \dots, \Delta_{N-n-1} \subset \mathbb{R}^N$, where Δ_i is the convex hull of the set of points A_J for all i-admissible sets J. The polyhedron Δ_i lies in the hyperplane*

$$x_1 + \cdots + x_N = i.$$

SOLUTION TO PROBLEM 3

Theorem. *The number of solutions of a generic system*

$$P_1 = \cdots = P_n = 0$$

in $\mathbb{P}^n \setminus \cup H_i$*, where* P_i *are Laurent polynomials in* $u_1, \ldots, u_N$*, whose Newton polyhedra* $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n \subset \mathbb{R}^N$ *lye in thee hyperplane* $x_1 + \cdots + x_N = 0$*, is equal to:*

$$m! V_m(\Delta_1, \Delta_2 - \Delta_1, \ldots, \Delta_{m-n} - \Delta_{m-n-1}, \tilde{\Delta}_1, \ldots, \tilde{\Delta}_n),$$

where $m = N - 1$ *and* $\Delta_1, \ldots, \Delta_{N-n-1}$ *is the characteristic sequence for the hyperplanes arrangement.*

Mixed volume in the formula makes sense, since all involved polyhedra belong to affine spaces which are parallel to the $(N-1)$-dimensional space $x_1 + \cdots + x_N = 0$.

LITERATURE

There are many papers dedicated to the Newton polyhedra theory. The most related to the lecture are the following papers:

[1] Khovanskii, A. *Newton polyhedra, and toroidal varieties.* Funct. Anal. Appl. 11 (1977), no. 4, 289–296 (1978).

[2] A. Khovanskii. *Newton Polyhedra and the genus of complete intersections.* Funct. Anal. Appl. 12 (1978), no. 1, 38–46 (1978).

[3] Danilov, V.I., & Khovanskii, A. *Newton polyhedra and an algorithm for computing Hodge–Deligne numbers.* Math. USSR – Izv. 29(1987), No. 2, 279–298.

The theory of vector valued Laurent polynomials was initiated very recently, see the following paper in preparation:

[4] Kaveh, K., Khovanskii A., and Spink, H. *Vector-valued Laurent polynomial equations, toric vector bundles and matroids.*