Name: ______________________________

SID: ______________________________

1. (4 points) True or False (fill in the blank with T or F)
   (a) An $m \times n$ matrix A has $Col(A) = \mathbb{R}^m$ if and only if it defines a surjective linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$. ☐
   (b) Suppose $T \colon V \to W$ is a linear transformation, $\underline{v}_1, \cdots, \underline{v}_n \in V$, such that $T\underline{v}_1, \cdots, T\underline{v}_n$ form a basis of W, then $\underline{v}_1, \cdots, \underline{v}_n$ form a basis of V. ☐
   (c) The rank of the transpose of a matrix equals the rank of the matrix itself. ☐
   (d) Two square matrices are similar if and only if they have the same characteristic polynomials. ☐

2. (4 points) True or False (fill in the blank with T or F)
   (a) $y'' + 4y = tan(t)e^{2t^2}$ can be solved using the method of undetermined coefficients. ☐
   (b) If A is 2×2 and has rank 1, then the dimension of the space of constant solutions (i.e. solutions of the form $\underline{x}(t) = \underline{v}$ for some constant vector v) to $\underline{x}' = A\underline{x}$ is 1. ☐
   (c) If A is 3×3 and diagonalizable then the matrix ODE $\underline{x}' = A\underline{x}$ is equivalent to three separate first order scalar ODEs. ☐
   (d) If A has no real eigenvalues then the matrix ODE $\underline{x}' = A\underline{x}$ has no real solutions. ☐

3. (4 points) Multiple choice: Which of these linear transformations is invertible? Mark an "x" in the box next to the correct answers:
   1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection about the line $2y = x$. ☐
   2. $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(\underline{e}_1) = \underline{e}_2$, $T(\underline{e}_2) = \underline{e}_3$. ☐
   3. $T : \mathbb{R}^3 \to \mathbb{R}^3$ with eigenvalues $-1, 0, 1$. ☐
   4. $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(\underline{x}) = A\underline{x}$ with $\det(A) = 0$. ☐

4. (4 points) Multiple choice: Choose all $\langle \underline{x}, \underline{y} \rangle$ that defines an inner product on $\mathbb{R}^3$. Mark an "x" in the box next to the correct answers:
   1. $\langle \underline{x}, \underline{y} \rangle = 2x_1y_1 + 2x_2y_2 + 2x_3y_3$. ☐
   2. $\langle \underline{x}, \underline{y} \rangle = x_1y_2 + x_2y_1 + x_3y_3$. ☐
   3. $\langle \underline{x}, \underline{y} \rangle = x_1x_2x_3 + y_1y_2y_3$. ☐
   4. $\langle \underline{x}, \underline{y} \rangle = 2x_1y_1 + x_2y_2 + x_3y_3 - x_1y_2 - x_2y_1$. ☐

5. (4 points) Multiple choice: Consider the function $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ defined by $f(x) = |sin(x)|$. Choose the correct value of the coefficients a_0, a_1 for the Fourier series for $f(x)$. [You may find this useful: $\sin(2x) = 2\sin(x)\cos(x)$]. Mark an "x" in the box next to the correct answers:

   1. $a_1 = 0$. ☐
   2. $a_0 = 2/\pi$. ☐
   3. $a_0 = 4/\pi$. ☐
   4. $a_1 = 2/\pi$. ☐

6. (6 points) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x - 2z \\ 5y \\ -2x + 4z \end{bmatrix}$

   Find a diagonal matrix D and a basis $\mathcal{B}$ of $\mathbb{R}^3$ so that the matrix for T relative to $\mathcal{B}$ is $[T]_{\mathcal{B}} = D$.

Additional space for problem 6:

7. (6 points) Find the standard matrix associated to the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ given by orthogonal projection onto the column space of

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

Additional space for problem 7:

8. (6 points) Let $\langle , \rangle$ be an inner products on $\mathbb{R}^n$ such that the associated quadratic form $Q(\underline{x}) = \langle \underline{x}, \underline{x} \rangle$ satisfies

$$Q(\underline{x}) = x_1^2 + x_2^2 + ... + x_n^2.$$

Show that $\langle \underline{x}, \underline{y} \rangle = \underline{x} \cdot \underline{y}$ for all $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^n$.

Additional space for problem 8:

9. (6 points) Consider the ODE

$$y'' - 4y = 0. \tag{1}$$

(a) (1 point) Find all real solutions $y(t)$ to (1).

(b) (2 points) Find all solutions to (1) satisfying the initial value condition

$$\begin{cases} y(-1) = 0 \\ y'(-1) = -4. \end{cases}$$

(c) (3 points) Find all real solutions to the nonhomogeneous ODE

$$y'' - 4y = t \sin 2t.$$

Additional space for problem 9:

10. (6 points) Solve the second order ODE

$$(t^2+1)x''(t) + 2(t+1)^2x'(t) + (t+1)(t+3)x(t) = 0. \tag{2}$$

(Hint: Consider $y(t) = (t^2+1)x(t)$. What ODE does $y(t)$ solve?)

Additional space for problem 10: