Math 250A
Professor Kenneth A. Ribet
Last Midterm Exam October 29, 2020
8:10–9:30AM

You have 80 minutes to work on the exam and 15 minutes to upload your work to `Gradescope`. You may consult the textbook, all the material on `bCourses`, the class `piazza` and your own notes. In case of questions, post a private note to instructors on `piazza`. Any clarifications or corrections that need to be promulgated will be added to a pinned post on `piazza`.

Not permitted: online searches, other uses of the internet, collaboration with other people (electronic or otherwise). Please act with honesty, integrity and respect for others.

1. Explain, citing examples and providing some detail, why:

a. Not all projective modules over a commutative ring are free.

b. Not all torsion-free abelian groups are free abelian groups.

c. Not all integral domains are unique factorization domains.

d. Not all unique factorization domains are principal ideal domains.

e. Every module over a ring is a quotient of a projective module over the ring.

2. Let A be the direct product $\prod_p \mathbf{Z}/p\mathbf{Z}$, taken over the set of prime numbers. Let $I \subset A$ be the direct sum $\bigoplus_p \mathbf{Z}/p\mathbf{Z}$.

a. Prove that the abelian group A/I is uniquely divisible, i.e., that multiplication by each positive integer is a bijection $A/I \xrightarrow{\sim} A/I$.

b. Show that I is an ideal of A. (View A as a ring via componentwise addition and multiplication.) Is I a prime ideal? Is it a maximal ideal?

3. Suppose that A is a Dedekind ring (Lang, p. 116). If A is a unique factorization domain, prove that A is a principal ideal domain. (At the conclusion of the exam is a *sketch* of a possible argument. You are welcome to adopt it, but then you'll need to fill in plenty of details.)

4. Let A be a commutative ring. An A-module M is *invertible* if there is an A-module N such that $M \otimes_A N$ is free of rank 1 over A. For each A-module M, let $[M]$ denote the isomorphism class of M.

a. Show that

$$[M] \cdot [N] := [M \otimes_A N]$$

gives a well-defined law of composition on the set of isomorphism classes of invertible A-modules and that this law of composition makes the set of isomorphism classes of invertible modules into an *abelian group*.

b. If A is a field, calculate the abelian group in part (a).

[Here is the promised sketch for problem 3: Every nonzero non-unit $\alpha \in A$ has a factorization as a product of prime elements, say $\alpha = \pi_1 \cdots \pi_t$. Because A is a UFD, the principal ideals (π_j) are all prime ideals. Meanwhile, (α) has a unique factorization $\mathfrak{p}_1 \cdots \mathfrak{p}_s$ as a product of prime ideals of A (Exercise 14, p. 116). A consequence is that all the prime ideals $\mathfrak{p}_i$ are principal. By varying the choice of α, we see that all prime ideals of A are principal and then that all ideals of A are principal.]

5. To finish, please copy and sign:

"*As a member of the UC Berkeley community, I acted with honesty, integrity, and respect for others during this exam. The work that I am uploading is my own work. I did not collaborate with or contact anyone during the exam. I did not obtain solutions from* `chegg.com` *or other sites. I adhered to all instructions for this examination.*"

Artwork Ig: `@itchyscabs` `@mimithemimo`