Math 10A final exam, December 16, 2016

1a. Find all points on the interval $[0,1]$ where the instantaneous rate of change of $f(x) = x^3 + x$ is equal to the average rate of change of $f(x)$ on the interval.

b. If the derivative of $f(x)$ is $\frac{1}{x^2+1}$, what is the derivative of $f(x^{-1})$?

2a. Suppose that n is a positive integer. Calculate the integral

$$\int_1^n \ln x \, dx.$$

b. For what values of x is the series $\sum_{n=1}^{\infty} \frac{n^2(x+7)^n}{10^n(n+1)^2}$ convergent?

3. What approximation to $(1.02)^{1/2}$ is provided by the quadratic Taylor polynomial for $f(x) = x^{1/2}$ at the point $a = 1$? (Leave your answer as an unsimplified numerical expression.)

4. Determine the volume of the solid obtained by revolving the area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 2$ about the x-axis.

5a. If a is a real number, calculate $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n$. (As for all problems on this exam, be sure to explain your reasoning with care.)

b. Let $f(x) = \frac{\ln x}{x}$ for $x > 0$. What happens to $f(x)$ as the positive number x approaches 0?

6a. Let $f(x)$ be the function $\frac{\ln x}{x}$, defined for x positive. Find $\lim_{x\to\infty} f(x)$.

b. Does $f(x)$ have a global maximum value? If so, what is this value?

7. Which is more likely: getting 60 or more heads in 100 tosses of a fair coin or getting 225 or more heads in 400 tosses of a fair coin?

As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.

8. If the continuous random variable X has PDF equal to $f(x)$, then we have

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

for all reasonable functions g. Use this information to calculate the expected value of $|X|$ when X is a standard normal variable (with mean 0 and standard deviation equal to 1).

9. Find all values of a and b such that

$$p(t) = \frac{ae^{bt}}{1 + ae^{bt}}$$

is a cumulative distribution function.

10. Explain how the approximation

$$\int_1^n \ln x\, dx \approx \ln(n!) - \frac{1}{2} \ln n$$

can be obtained by averaging together left- and right-endpoint approximations to the integral. (Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$.)

Problems 2a and 10 lead to *Stirling's approximation* to $n!$.

Thanks everyone for coming together to make Math 10A a great class. I look forward to seeing you next semester—and beyond—in Evans, at the Faculty Club, in the RSF, on Yelp and on Facebook. Have wonderful winter break!