George M. Bergman Spring 2001, Math 53M 14 May, 2001

Bechtel Auditorium **Final Examination** 8:00-11:00 AM

1. (60 points, 6 points apiece) Find the following. If an expression is undefined, say so.

(a) The area of the region of the plane described in polar coordinates by the conditions $0 \leq \theta \leq 1$, $0 \leq r \leq 1 + e^{\theta}$.

(b) A unit vector perpendicular to both $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 6 \rangle$.

(c) The length of the curve given by $x = t^2$, $y = (t^3/3) - t$, where $-1 \leq t \leq 1$.

(d) $\frac{d}{dt} f(g(t^2), g(t^3))$, where f is a differentiable function of two variables and g is a differentiable function of one variable. The answer should be expressed in terms of f, g, and their derivatives and/or partial derivatives.

(e) $\int_0^1 \int_{1-x}^{1+x} xy\,dy\,dx$.

(f) An expression for $\iiint_E f(x, y, z)\,dV$, as an iterated integral, where $E = \{ (x, y, z) \mid x^2 + y^2 + z^2 \leq 100 \}$, and f is a continuous function. (Do not change coordinates.)

(g) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field $\langle 1, 2, y \rangle$ and C is the curve given by $\mathbf{r}(t) = \langle t^2, t^3, t^5 \rangle$ for $0 \leq t \leq 10$.

(h) An expression for $\iint_{D_1} f(x, y)\,dx\,dy$ as an integral over D_2, where D_1 is a region of the x-y-plane, and D_2 is a region of the u-v-plane which is mapped in a one-to-one fashion onto D_1 by the transformation $x = uv$, $y = u^2/v$; and where f is a continuous function on D_1.

(i) $\int_C xy^{-1}\,dx$, where C is the segment of the curve $y = x^3$ between $x = -1$ and $x = 1$.

(j) $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}$ is the constant vector field $\langle 2, 1, -1 \rangle$, and S is the parallelogram with vertices $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ and $(1, -2, 3)$, and upward orientation.

2. (12 points; 4 points each) In each part, give the definition asked for. Note that you are *not* asked to give examples or other related information.

(a) Define the partial derivative $f_1(a,b)$ of a function f at a point (a,b) of its domain. (There are many other symbols for this, e.g., $(\partial f/\partial x)(a,b)$, but we are asking for a definition, not alternative notation. A brief general statement of how one would find this partial derivative would be an acceptable answer.)

(b) What does it mean to say that a function f of two variables is *differentiable*?

(c) If f is a function of two variables, and $\mathbf{u}$ a unit vector in the plane, what is meant by the *directional derivative* $D_{\mathbf{u}} f$?

3. (12 points) (a) (6 points) Find constants a and b such that the vector field $\langle 3x^2y\sin y - 2x^2\cos y,\ ax^3y\cos y + bx^3\sin y\rangle$ is the gradient of a function $f(x,y)$. (You do not have to find the function f.)

(b) (6 points) Suppose a and b are as in part (a), so that $\mathbf{F}$ is the gradient of a function. Prove that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every curve C beginning at $(0,0)$ and ending at $(0,1)$. (Suggestion: first show this fact for one particular curve. You do not have to have done part (a) to do part (b).)

4. (8 points) Let S_1 be the hemisphere $z = \sqrt{(1-x^2-y^2)}$ and S_2 the hemisphere $z = -\sqrt{(1-x^2-y^2)}$, both with upward orientation, and let E be the solid ball $\{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$. If $\mathbf{F}$ is any vector field whose component functions have continuous partial derivatives on an open set containing E, write an equation expressing the triple integral over E of the divergence of $\mathbf{F}$ in terms of surface integrals over S_1 and S_2.

5. (8 points) Let $\mathbf{F}$ be the vector field $\langle xy,\ yz^2,\ zx^3\rangle + \nabla e^{x+\sin y}$, and let S be the part of the surface $z = xy^2(1-x-y)^3$ lying above the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$, with upward orientation. Compute $\iint_S \operatorname{curl} \mathbf{F}\cdot d\mathbf{S}$. (Suggestion: Treat the two summands of $\mathbf{F}$ separately.)